SM 316 – Spring 2019

Homework 4

Due: Monday 11 FEB 2019

PLEASE READ THE INSTRUCTIONS/SUGGESTIONS ON THE COURSE WEBPAGE.

Hand in the following problems:

1. From the text book, 4.2, 4.12, 4.20, 4.34.

   4.2: The probability distribution of the discrete random variable X is

$$f(x) = \binom{3}{x}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{3-x}, \quad x = 0, 1, 2, 3.$$

   Find the mean of X.

   Solution: *The mean of X is*

$$E[X] = \sum_{\text{all } x} x f(x) = \sum_{x=0}^{3} x \binom{3}{x}\left(\frac{1}{4}\right)^{x}\left(\frac{3}{4}\right)^{3-x},$$

$$= 0 + 1 \cdot \binom{3}{1}\left(\frac{1}{4}\right)^{1}\left(\frac{3}{4}\right)^{2} + 2 \cdot \binom{3}{2}\left(\frac{1}{4}\right)^{2}\left(\frac{3}{4}\right)^{1} + 3 \cdot \binom{3}{3}\left(\frac{1}{4}\right)^{3}\left(\frac{3}{4}\right)^{0} = \frac{3}{4}$$

   4.12: If a dealers profit, in units of \$5000, on a new automobile can be looked upon as a random variable X having the density function

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

   find the average profit per automobile.

   Solution: *By definition, the average is*

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{0}^{1} x[2(1-x)]\, dx = \frac{1}{3}.$$

   Since the units are in \$5000, then the average profit is

$$\textit{Average profit} = \$\frac{5000}{3}$$

4.20: A continuous random variable X has the density function

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = e^{2X/3}$.

Solution: *By definition we have*

$$E[e^{2X/3}] = \int_{-\infty}^{\infty} e^{2x/3} f(x)\, dx = \int_{0}^{\infty} e^{2x/3} e^{-x}\, dx$$
$$= \int_{0}^{\infty} e^{-x/3}\, dx = \left(-3e^{-x/3}\right)_0^{\infty} = 3.$$

4.34: Let X be a random variable with the following probability distribution:

x	-2	3	5
$f(x)$	0.3	0.2	0.5

Find the standard deviation of X.

Solution: *Note the standard deviation is the square root of the variance of X. First we must find the mean.*

$$E[X] = \sum_{all\ x} x f(x) = -2 \cdot 0.3 + 3 \cdot 0.2 + 5 \cdot 0.5 = 2.5.$$

The variance is

$$E[(X-\mu)^2] = \sum_{all\ x} (x-2.5)^2 f(x) = (-2-2.5)^2 \cdot 0.3 + (3-2.5)^2 \cdot 0.2 + (5-2.5)^2 \cdot 0.5 = 9.25.$$

Thus the standard deviation of X is

$$\sigma_X = \sqrt{E[(X-\mu)^2]} = 3.041.$$

2. Suppose that X is a random variable with mean 2 and variance 3.

   (a) Compute $E(X-1)^2$.

   Solution: *We foil the expected value to obtain*

   $$E[(X-1)^2] = E[X^2 - 2X + 1] = E[X^2] - 2E[X] + 1.$$

   We know the mean is 2 and the variance is 3. We can find $E[X^2]$ by noting

   $$Var(X) = E[(X-\mu)^2] = E[X^2 - 2\mu X + \mu^2] = E[X^2] - 2\mu^2 + \mu^2$$
   $$= E[X^2] - \mu^2.$$

Thus

$$Var(X) = 3 = E[X^2] - (2)^2$$
$$7 = E[X^2].$$

Therefore,

$$E[(X-1)^2] = E[X^2] - 2E[X] + 1 = 7 - 2 \cdot 2 + 1 = 4.$$

(b) Compute Var$(2X + 1)$.
Solution: *By definition*

$$Var(2X+1) = E[(2X + 1 - E[2X+1])^2] = E[(2X - 2E[X])^2]$$
$$= 4E[(X-\mu)^2] = 4\,Var(X) = 4 \cdot 3 = 12.$$

3. Consider these two random variables X and Y,

$$P(X=0) = \frac{1}{2}, \quad P(X=1) = P(X=-1) = \frac{1}{4},$$

and

$$P(Y=0) = \frac{1}{2}, \quad P(Y=10) = P(Y=-10) = \frac{1}{4}.$$

Find $E[X]$ and $E[Y]$. Next find the variance of X and variance of Y. Discuss why the mean of X and Y don't tell the whole story about the random variables.

Solution: *The mean of each is*

$$E[X] = \sum_{all\ x} xf(x) = -1\frac{1}{4} + 0\frac{1}{2} + 1\frac{1}{4} = 0,$$

and

$$E[Y] = \sum_{all\ y} yf(y) = -10\frac{1}{4} + 0\frac{1}{2} + 10\frac{1}{4} = 0.$$

The variance is

$$Var(X) = E[(X-\mu)^2] = E[X^2] = (-1)^2\frac{1}{4} + 0^2\frac{1}{2} + (1)^2\frac{1}{4} = \frac{1}{2},$$

and

$$Var(Y) = E[(Y-\mu)^2] = E[Y^2] = (-10)^2\frac{1}{4} + 0^2\frac{1}{2} + (10)^2\frac{1}{4} = 50.$$

Thus the variance of Y is much higher than X. We can imagine X is a game where we win \$1 with probability 0.25, lose \$1 with probability 0.25 and tie with probability 0.5. Y is a similar game where we lose or win \$10. Both games are mean 0, but Y you are risking 10 times as much much (with the chance of a $\times 10$ payout).

4. The length of time, in minutes, for an airplane to obtain clearance for takeoff at a certain airport is a random variable $Y = 3X - 2$, where X has the density function

$$f(x) = \begin{cases} \frac{1}{4}e^{-x/4}, & x > 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean and variance of the random variable Y.

Solution: *First we find the mean of X*

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{0}^{\infty} \frac{x}{4} e^{-x/4}\, dx = 4,$$

where we had to perform integration by parts. The variance of X is,

$$Var(X) = \int_{-\infty}^{\infty} (x-4)^2 f(x)\, dx = \int_{0}^{\infty} \frac{(x-4)^2}{4} e^{-x/4}\, dx = 16,$$

where we did integration by parts twice. Then the mean of Y is

$$E[Y] = E[3X - 2] = 3E[X] - 2 = 3 \cdot 4 - 2 = 10 \textit{ minutes to obtain clearance.}$$

The variance is

$$Var(Y) = E[(3X - 2 - E[3X - 2])^2] = 9E[(X - \mu)^2] = 9\,Var(X)$$

$$= 9 \cdot 16 = 144.$$