SM 316 – Spring 2019

Homework 5
Solutions

Due: Monday 04 MAR 2019

PLEASE READ THE INSTRUCTIONS/SUGGESTIONS ON THE COURSE WEBPAGE. FOR ALL PROBLEMS, MAKE SURE TO DEFINE WHAT DISTRIBUTION YOU ARE USING AND USE CORRECT NOTATION.

Hand in the following problems:

1. A student must get at least three of the four problems on the exam correct to get an A. The student has been able to do 80% of the examples on old exams so they assume the probability to get any question correct is 0.8. Assume that the results on different problems are independent.

   (a) What is the probability the student gets an A?

   Solution: *Let X the the number of problems the student gets right out of the 4 question test. Then X is a binomial distribution with $n = 4$, $p = 0.8$. The event that the student gets an A is $\{X \geq 3\}$. Thus*

$$P(X \geq 3) = P(X = 3) + P(X = 4) = \binom{4}{3}(0.8)^3(0.2) + \binom{4}{4}(0.8)^4(0.2)^0$$

$$= 0.4096 + 0.4096 = 0.8192$$

   (b) If the student gets the first problem correct, what is the probability the student gets an A?

   Solution: *Here we want to find $P(X \geq 3|$ 1st question is correct). Let Y be the random variable of the number of questions that the student gets correct on questions 2, 3, and 4 of the test. Then we want to find $\{Y \geq 2\}$. Note that Y is a binomial random variable with $n = 3$ and $p = 0.8$. Because the questions are independent we have*

$$P(X \geq 3|\textit{1st question is correct}) = \frac{P(Y \geq 2, \textit{1st question is correct})}{P(\textit{ 1st question is correct})}$$

$$= \frac{P(Y \geq 2)P(\textit{1st question is correct})}{P(\textit{1st question is correct})} = P(Y = 2) + P(Y = 3)$$

$$= \binom{3}{2}(0.8)^2(0.2) + \binom{3}{3}(0.8)^3(0.2)^0 = 0.384 + 0.512 = 0.896$$

2. A certain area of the Midwest is, on average, hit by 6 tornados a year. Find the probability that in a given year that area will be hit by

   (a) fewer than 4 tornados.

   Solution: *Let X be the number of tornados that hit the certain area of the Midwest in the time period t. X is a Poisson random variable with parameters $\lambda = 6$ and time t. For $t = 1$ we want to find*

   $$P(X < 4) = P(X \leq 3).$$

   This is the cumulative distribution function (CDF) at $x = 3$. We can use the calculator function POISSONCDF with ... to obtain

   $$P(X \leq 3) \approx 0.1512$$

   (b) anywhere from 6 to 8 tornados.

   Solution: *Here we want to find $P(6 \leq X \leq 8)$. We can do this in two different ways. One by using the probability mass function (PMF) POISSONPDF (3 times)*

   $$P(6 \leq X \leq 8) = P(X = 6) + P(X = 7) + P(X = 8) = 0.1606 + 0.1377 + 0.1033 = 0.4016$$

   or using the CDF (2 times)

   $$P(6 \leq X \leq 8) = P(X \leq 8) - P(X \leq 5) = 0.8472 - 0.4457 = 0.4015.$$

3. Let X be a continuous random variable with Uniform distribution on (10,20) which represents the time, in years, that an electrical component fails.

   (a) Find the probability the component will last more than 16 years given the component has lasted 12 years thus far.

   Solution: *Note that the probability density function (pdf) for X is*

   $$f(x) = \begin{cases} \frac{1}{20-10}, & 10 < x < 20 \\ 0, & \textit{otherwise.} \end{cases}$$

   Then the conditional probability is

   $$P(X > 16 | X > 12) = \frac{P(X > 16, X > 12)}{P(X > 12)}.$$

   Note that if $X > 16$ and $X > 12$ then X is just greater than 16. So

   $$P(X > 16 | X > 12) = \frac{P(X > 16, X > 12)}{P(X > 12)} = \frac{P(X > 16)}{P(X > 12)}.$$

   Now we can compute the probabilities using integrals,

   $$P(X > 16 | X > 12) = \frac{P(X > 16)}{P(X > 12)} = \frac{\int_{16}^{20} \frac{1}{10}\, dx}{\int_{12}^{20} \frac{1}{10}\, dx} = \frac{1}{2}.$$

(b) Find the cumulative distribution function $F(x)$.

Solution: *The cumulative distribution function (CDF) is defined as*

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt.$$

Note that the PDF is split into three intervals $(-\infty, 10)$, $(10, 20)$ *and* $(20, \infty)$. *Thus*

$$F(x) = \begin{cases} \int_{-\infty}^{x} 0\, dt & = 0 & -\infty < x \leq 10 \\ \int_{-\infty}^{10} 0\, dt + \int_{10}^{x} \frac{1}{10}\, dt & = \frac{x}{10} - 1 & 10 < x \leq 20 \\ \int_{-\infty}^{10} 0\, dt + \int_{10}^{20} \frac{1}{10}\, dt + \int_{20}^{x} 0\, dt & = 1 & 20 < x \end{cases}$$

(c) Use part b) to find the probability the component lasts less than 18 years.

Solution: *We wish to find* $P(X \leq 18)$. *Using the middle column of the CDF* $F(x)$ *above we find*

$$P(X \leq 18) = F(18) = \frac{18}{10} - 1 = 0.8.$$

4. The loaves of rye bread distributed to local stores by a certain bakery have an average length of 30 centimeters and a standard deviation of 2 centimeters. Assuming that the lengths are normally distributed, what percentage of the loaves are

(a) longer than 31.7 centimeters?

Solution: *Let* X *be the length of the bread. Then* X *is normally distributed with mean* $\mu = 30$ *and standard deviation* $\sigma = 2$. *Using our calculator NORMALCDF, we can find*

$$P(X > 31.7) \approx 0.1977$$

(b) between 29.3 and 33.5 centimeters in length?

Solution: *Again, using our calculators with NORMALCDF,*

$$P(29.3 < X < 33.5) \approx 0.5968.$$

(c) Are shorter than 25.5 centimeters?

Solution: *Again, using our calculators with NORMALCDF,*

$$P(X < 25.5) \approx 0.0122.$$

(d) Is it possible to have positive probability for a negative length? Why or why not?
Solution: *Yes it is possible. Note that the density for a normal random variable,*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},$$

is defined for all real numbers and $f(x) > 0$ *for all* x*. Thus*

$$P(X < 0) = \int_{-\infty}^{0} f(x)\, dx > 0.$$

However, using our calculators shows that this probability is ver small (smaller than the accuracy of the calculator. Thus we can safely assume that this situation wont arise in our lifetime.

5. Let X be a normal random variable with mean 3 and variance 4.

   (a) Find the probability $P(2 < X < 6)$.

   Solution: *Again, using our calculators with NORMALCDF,*

   $$P(2 < X < 6) \approx 0.6247.$$

   (b) Find the value k such that $P(X > k) = 0.33$.

   Solution:*Here we need to use the table. To do so, we must change* X *to a mean 0, standard deviation 1 Normal random variable* ($N(0,1)$)*. To do this we note*

   $$Z = \frac{X - \mu}{\sigma}$$

   is a zero mean, standard deviation 1 random variable. First note that to use the CDF we have

   $$P(X > k) = 1 - P(X \leq k) = 0.33$$

   or

   $$P(X \leq k) = 0.67$$

   . Transforming to a $N(0,1)$*,*

   $$P(X \leq k) = P(X - 3 \leq k - 3) = P(\frac{X-3}{2} \leq \frac{k-3}{2}) = 0.67$$

   since $\sigma^2 = 4$ *and* $\sigma = 2$*. The table gives,*

   $$P(Z \leq 0.44) = 0.6700.$$

   Thus

   $$0.44 = \frac{k-3}{2}$$

   or

   $$k = 3.88.$$

   (c) Find $E[X^2]$. (Hint: You can integrate with the density function, but it is quicker to relate $E[X^2]$ with the mean and variance)

   Solution: *Note that the variance is*

   $$Var(X) = E[(X-\mu)^2] = E[X^2] - \mu^2 = 4.$$

   Thus

   $$E[X^2] = 4 + \mu^2 = 4 + (3)^2 = 13.$$