SM 316 – Spring 2019

Homework 6
Solutions

Due: Monday 20 MAR 2019

Please read the instructions/suggestions on the course webpage.

Hand in the following problems:

1. The length of time for one individual to be served at a cafeteria is a random variable having an exponential distribution with a mean of 4 minutes. What is the probability that a person is served in less than 3 minutes?

   Solution: *Let X be the length of time for one individual to be served at a cafeteria. The mean of X is*
   $$E[X] = 4.$$
   Since X is exponential, and $E[X] = \beta$ for an exponential, then $\beta = 4$. We want to find $P(X < 3)$. Using the probability density function of X, an exponential random variable with $\beta = 4$, we have
   $$P(X < 3) = \int_0^3 \frac{1}{4} e^{-x/4}\, dx = -e^{-x/4}\Big|_{x=0}^{x=3} = 1 - e^{-3/4}.$$

2. The life, in years, of a certain type of electrical switch has an exponential distribution with an average life $\beta = 2$. what is the probability it fails during the first year?

   Solution: *This problem is very similar to the previous problem. Let X be the life, in year, of the certain type of electrical switch. X is an exponential random variable with $\beta = 2$. We want to find $P(X < 1)$. Using the pdf of an exponential random variable with $\beta = 2$, we have*
   $$P(X < 1) = \int_0^1 \frac{1}{2} e^{-x/2}\, dx = -e^{-x/2}\Big|_{x=0}^{x=1} = 1 - e^{-1/2}.$$

3. The lengths of time, in minutes, that 10 patients waited in a doctor's office before receiving treatment were recorded as follows: 5, 11, 9, 5, 10, 15, 6, 10, 5, and 10. Treating the data as a random sample, find

   (a) the sample mean;

   Solution: *The sample mean is,*
   $$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{10}(5 + 11 + 9 + 5 + 10 + 15 + 6 + 10 + 5 + 10) = 8.6$$

(b) the sample mode;

Solution: *The sample mode is the most frequent occurrence. Since both 5 and 10 happen three times, then the sample mode is 5 and 10.*

(c) the sample median;

Solution: *To find the sample median, we first must order the data from smallest to largest. So*

$$x_i = \{5, 5, 5, 6, 9, 10, 10, 10, 11, 15\}.$$

There are an even number of data (10), thus by the definition of median, we have

$$\tilde{X} = \frac{1}{2}(x_5 + x_6) = \frac{1}{2}(9 + 10) = 9.5.$$

(d) the sample variance;

Solution: *The formula for sample variance is*

$$S^2 = \frac{1}{(n-1)}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{9}((5-8.6)^2 + (11-8.6)^2 + (9-8.6)^2 + (5-8.6)^2 + (10-8.6)^2$$

$$+(15 - 8.6)^2 + (6 - 8.6)^2 + (10 - 8.6)^2 + (5 - 8.6)^2 + (10 - 8.6)^2) = 10.93$$

4. If the standard deviation of the mean for the sampling distribution of random samples of size 36 from a large or infinite population is 2, how large must the sample size become if the standard deviation is to be reduced to 1.2?

Solution: *Define $X_1, X_2, ..., X_n$ to be our random samples. Define*

$$E[X_i] = \mu, \quad \sigma = \sqrt{Var(X_i)},$$

To be the true mean and standard deviation, respectively, for the population. From class we showed that for the sample mean

$$\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n)$$

, the mean and the standard deviation for the sample mean are

$$E[\bar{X}] = \mu, \quad \sigma_{\bar{X}} = \sqrt{Var(\bar{X})} = \frac{\sigma}{\sqrt{n}}.$$

The problem states that "the standard deviation of the mean for the sampling distribution of random samples of size 36...is 2." We take that to mean

$$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{36}} = 2.$$

Solving for σ gives $\sigma = 12$. With a different sample of size n_1, how do we reduce the sample standard deviation to 1.2? That is, we want to find the n_1 such that

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n_1}} = \frac{12}{\sqrt{n_1}} = 1.2.$$

Solving for n_1 gives,

$$n_1 = 100.$$

5. A soft-drink machine is regulated so that the amount of drink dispensed averages 240 milliliters with a standard deviation of 15 milliliters. Periodically, the machine is checked by taking a sample of 40 drinks and computing the average content. If the mean of the 40 drinks is a value within the interval $\mu_{\bar{X}} \pm 2\sigma_{\bar{X}}$, the machine is thought to be operating satisfactorily; otherwise, adjustments are made. The company official found the mean of 40 drinks to be $\bar{X} = 236$ milliliters and concluded that the machine needed no adjustment. Was this a reasonable decision?

   Solution: *We want to check if 236 is within two standard deviations (of the sample mean!) of the sample mean of 240. From the work done in the last problem, the standard deviation of the sample mean is*

   $$\sigma_{\bar{X}} = \frac{\sigma}{\sqrt{n}} = \frac{15}{\sqrt{40}} \approx 2.3717.$$

   Thus 236 is just within $(235.2566, 244.7434)$ with is within two standard deviations of the sample mean.

6. If a certain machine makes electrical resistors having a mean resistance of 40 ohms and a standard deviation of 2 ohms, what is the probability that a random sample of 36 of these resistors will have a combined resistance of more than 1458 ohms?

   Solution: *To use the central limit theorem, we must change the problem to ask a question about the sample mean $\bar{X}$. If the combined resistance of the samples is more than 1458, then the sample mean is*

   $$\bar{X} > \frac{1458}{36} = 40.5.$$

   Note the central limit theorem states,

   $$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

   is approximately a standard normal random variable. Thus we want to find,

   $$P(\bar{X} > 40.5) = P\left(\frac{\bar{X} - 40}{2/\sqrt{36}} > \frac{40.5 - 40}{2/\sqrt{36}}\right) = P(Z > 1.5)$$

   where Z is a standard normal RV. Using the calculator NormalCDF we have

   $$P(\bar{X} > 40.5) = P(Z > 1.5) = 0.0668.$$