SM 316 – Spring 2019

Homework 7
Solutions

Due: Wednesday 27 MAR 2019

PLEASE READ THE INSTRUCTIONS/SUGGESTIONS ON THE COURSE WEBPAGE.

Hand in the following problems:

1. Do the following problems from the textbook: 8.37, 8.39, 8.41, 8.45, 8.48, 9.2, 9.3, 9.11

8.37: For a chi-squared distribution, find (a) $\chi^2_{0.025}$ when $v = 15$.

Solution: *Using the table for* $\alpha = 0.025$ *and* $v = 15$ *yields*

$$\chi^2_{0.025} = 27.488.$$

(b) $\chi^2_{0.01}$ when $v = 7$.

Solution: *Using the table for* $\alpha = 0.01$ *and* $v = 7$ *yields*

$$\chi^2_{0.01} = 18.475.$$

(c) $\chi^2_{0.05}$ when $v = 24$.

Solution: *Using the table for* $\alpha = 0.05$ *and* $v = 24$ *yields*

$$\chi^2_{0.05} = 36.415.$$

8.39: For a chi-squared distribution, find χ^2_α such that

(a) $P(X^2 > \chi^2_\alpha) = 0.99$ when $v = 4$.

Solution: *Here we look at* $\alpha = 0.99$ *and* $v = 4$ *to find*

$$P(X^2 > 0.297) = 0.99.$$

So

$$\chi^2_\alpha = \chi^2_{0.99} = 0.297 \text{ for } v = 4.$$

(b) $P(X^2 > \chi^2_\alpha) = 0.025$ when $v = 19$.

Solution: *Here we look at* $\alpha = 0.025$ *and* $v = 19$ *to find*

$$P(X^2 > 32.852) = 0.025.$$

So

$$\chi^2_\alpha = \chi^2_{0.025} = 32.852 \text{ for } v = 19.$$

(c) $P(37.652 < X^2 < \chi^2_\alpha) = 0.045$ when $v = 25$.

Solution: *Here we rewrite the inequality in a way that we can use the table. Note that*

$$P(\chi^2_\beta < X^2 < \chi^2_\alpha) = P(X^2 > \chi^2_\beta) - P(X^2 > \chi^2_\alpha).$$

So

$$P(37.652 < X^2 < \chi^2_\alpha) = P(X^2 > 35.652) - P(X^2 > \chi^2_\alpha) = 0.045.$$

First we look at the table for $v = 25$ *and identify the* χ^2_α *value for* 37.652 *and find*

$$\chi^2_{0.05} = 37.652 \text{ for } v = 25.$$

Thus

$$0.05 - P(X^2 > \chi^2_\alpha) = 0.045$$

or

$$P(X^2 > \chi^2_\alpha) = 0.005.$$

Now we use the table for $\alpha = 0.005$ *and* $v = 25$ *to find*

$$\chi^2_{0.005} = 46.928 \text{ for } v = 25.$$

8.41: Assume the sample variances to be continuous measurements. Find the probability that a random sample of 25 observations, from a normal population with variance $\sigma^2 = 6$, will have a sample variance S^2

(a) greater than 9.1;

Solution: *Here we want to find*

$$P(S^2 > 9.1).$$

First we must change the probability to that involving the chi-squared distribution. That is,

$$X^2 = \frac{(n-1)S^2}{\sigma^2}$$

has chi-squared distribution with $v = (n-1)$. *Thus*

$$P(S^2 > 9.1) = P\left(\frac{(n-1)}{\sigma^2}S^2 > \frac{(n-1)}{\sigma^2}9.1\right) = P\left(\frac{24}{6}S^2 > \frac{24}{6}9.1\right)$$

$$= P(X^2 > 36.4).$$

Looking up 36.4 in the chi-squared table for $v = 24$ yields,

$$\chi^2_{0.05} = 36.415 \approx 36.4 \textit{ for } v = 25.$$

Thus

$$P(S^2 > 9.1) = P(X^2 > 36.4) = 0.05.$$

(b) between 3.462 and 10.745.

Solution: *Using our nifty trick above from problem 8.39 c) we can write*

$$P(3.462 < S^2 < 10.745) = P(S^2 > 3.462) - P(S^2 > 10.745).$$

Now we change both probabilities to χ^2 statistics using the technique from part a) with $n-1 = 24$ and $\sigma^2 = 6$.

$$P(3.462 < S^2 < 36.4) = P(S^2 > 3.462) - P(S^2 > 36.4)$$
$$= P\left(\frac{24}{6}S^2 > \frac{24}{6}3.462\right) - P\left(\frac{24}{6}S^2 > \frac{24}{6}10.745\right)$$
$$= P(X^2 > 13.848) - P(X^2 > 42.980).$$

We look up these values in the chi-squared table for $v = 24$ to find

$$\chi^2_{0.95} = 13.848, \textit{ and } \chi^2_{0.01} = 42.980 \textit{ for } v = 24.$$

Thus

$$P(3.462 < S^2 < 36.4) = 0.95 - 0.01 = 0.94.$$

8.45:

(a) Find $P(T < 2.365)$ when $v = 7$.

Solution: *To use the table, we must transform the inequality to,*

$$P(T < 2.365) = 1 - P(T > 2.365).$$

Using the table for $v = 7$ we see,

$$t_{0.025} = 2.365, \textit{ for } v = 7.$$

Thus

$$P(T < 2.365) = 1 - 0.025 = 0.975.$$

(b) Find $P(T > 1.318)$ when $v = 24$.

Solution: *Using the table directly for $v = 24$ yields,*

$$t_{0.10} = 1.318,$$

and thus

$$P(T > 1.318) = 0.10.$$

(c) Find $P(-1.356 < T < 2.179)$ when $v = 12$.

Solution:*Here we use symmetry for the fact that*

$$P(T < -t_\alpha) = P(T > t_\alpha).$$

Thus we can write,

$$P(-1.356 < T < 2.179) = 1-P(-1.356 > T)-P(T > 2.179) = 1-P(T > 1.356)-P(T > 2.179).$$

Looking up these values in the table for $v = 12$ *yields*

$$t_{0.10} = 1.356, \text{ and } t_{0.025} = 2.179, \text{ for } v = 12.$$

Thus

$$P(-1.356 < T < 2.179) = 1 - 0.10 - 0.025 = 0.875.$$

(d) Find $P(T > -2.567)$ when $v = 17$.

Solution: *Again, so we can use the table we write*

$$P(T > -2.567) = 1 - P(T < -2.567).$$

Then symmetric gives,

$$P(T > -2.567) = 1 - P(T < -2.567) = 1 - P(T > 2.567).$$

The table yields

$$t_{0.01} = 2.567, \text{ for } v = 17.$$

8.48: A manufacturing firm claims that the batteries used in their electronic games will last an average of 30 hours. To maintain this average, 16 batteries are tested each month. If the computed t-value falls between $-t_{0.025}$ and $t_{0.025}$, the firm is satisfied with its claim. What conclusion should the firm draw from a sample that has a mean of $\bar{x} = 27.5$ hours and a standard deviation of $s = 5$ hours? Assume the distribution of battery lives to be approximately normal.

Solution: *Here we use the fact that the true standard deviation* σ *is unknown. Thus we know*

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

has a t*-distribution with* $v = n - 1$ *degrees of freedom. Looking at the* t*-distribution table for* $v = 16 - 1 = 15$ *gives*

$$t_{0.025} = 2.131, \text{ for } v = 15.$$

Thus we want to know if T satisfies

$$-2.131 < T < 2.131.$$

Computing T gives,

$$T = \frac{27.5 - 30}{5/\sqrt{16}} = -2.$$

Since

$$-2.131 < -2 < 2.131$$

is just barely satisfied, then the firm should be satisfied with its claim.

9.2: An electrical firm manufactures light bulbs that have a length of life that is approximately normally distributed with a standard deviation of 40 hours. If a sample of 30 bulbs has an average life of 780 hours, find a 96% confidence interval for the population mean of all bulbs produced by this firm.

Solution: *Since the population value of $\sigma = 40$ is known, we want to use the confidence interval for σ known. Thus the $1 - \alpha$ confidence interval is*

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where $z_{\alpha/2}$ is the value from the Standard Normal distribution table for

$$P(Z > z_{\alpha/2}) = \alpha/2.$$

For a 96% confidence interval we have $\alpha = 0.04$ and $\alpha/2 = 0.02$. Using the table, we find a number close to 0.02 in the table and thus,

$$z_{0.02} \approx z_{0.0202} = -2.05.$$

We can drop the negative because symmetry gives

$$P(Z < -2.05) = P(Z > 2.05) = 0.0202.$$

Thus $z_{0.02} \approx 2.05$. Thus the 96% confidence interval is

$$780 - (2.05)\frac{40}{\sqrt{30}} < \mu < 780 + (2.05)\frac{40}{\sqrt{30}}$$

$$765.03 < \mu < 794.97.$$

9.3: Many cardiac patients wear an implanted pacemaker to control their heartbeat. A plastic connector module mounts on the top of the pacemaker. Assuming a standard deviation of 0.0015 inch and an approximately normal distribution, find a 95% confidence interval for the mean of the depths of all connector modules made by a certain manufacturing company. A random sample of 75 modules has an average depth of 0.310 inch.

Solution: *Since the population value of* $\sigma = 0.0015$ *is known, we want to use the confidence interval for* σ *known. Thus the* $1-\alpha$ *confidence interval is*

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where $z_{\alpha/2}$ *is the value from the Standard Normal distribution table for*

$$P(Z > z_{\alpha/2}) = \alpha/2.$$

For a 95% confidence interval we have $\alpha = 0.05$ *and* $\alpha/2 = 0.025$*. Using the table, we find a number close to 0.025 in the table and thus,*

$$z_{0.025} = -1.96.$$

We can drop the negative because symmetry gives

$$P(Z < -1.96) = P(Z > 1.96) = 0.0250.$$

Thus $z_{0.025} = 1.96$*. Thus the 95% confidence interval is*

$$0.310 - (1.96)\frac{0.0015}{\sqrt{75}} < \mu < 0.310 + (1.96)\frac{0.0015}{\sqrt{75}}$$
$$0.3097 < \mu < 0.3103.$$

9.11: A machine produces metal pieces that are cylindrical in shape. A sample of pieces is taken, and the diameters are found to be 1.01, 0.97, 1.03, 1.04, 0.99, 0.98, 0.99, 1.01, and 1.03 centimeters. Find a 99% confidence interval for the mean diameter of pieces from this machine, assuming an approximately normal distribution.

Solution: *Since the population value of* σ *is unknown, we want to use the confidence interval for* σ *unknown. Thus the* $1-\alpha$ *confidence interval is*

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ *is the value from the t-distribution table for*

$$P(T > t_{\alpha/2}) = \alpha/2.$$

For a 99% confidence interval we have $\alpha = 0.01$ *and* $\alpha/2 = 0.005$*. Using the table for* $v = 9-1 = 8$ *yields*

$$t_{0.005} = 3.355, \text{ for } v = 8.$$

The calculated sample mean and variance are

$$\bar{x} = 1.0056, \quad s^2 = 0.00060278 \implies s = \sqrt{s^2} = 0.0246.$$

Thus the 99% confidence interval is

$$1.0056 - (3.355)\frac{0.0246}{\sqrt{9}} < \mu < 1.0056 + (3.355)\frac{0.0246}{\sqrt{9}}$$
$$0.9781 < \mu < 1.0331.$$

2. A random sample of 100 automobile owners in the state of Virginia shows that an automobile is driven on average 23,500 kilometers per year with a standard deviation of 3900 kilometers. Assume the distribution of measurements to be approximately normal.

   (a) Construct a 99% confidence interval for the average number of kilometers an automobile is driven annually in Virginia.

   Solution: *Since the population value of σ is unknown, we want to use the confidence interval for σ unknown. Thus the $1-\alpha$ confidence interval is*

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

   where $t_{\alpha/2}$ is the value from the t-distribution table for

$$P(T > t_{\alpha/2}) = \alpha/2.$$

   For a 99% confidence interval we have $\alpha = 0.01$ and $\alpha/2 = 0.005$. Using the table for $v = 120$, since 120 is close to $v = 100 - 1 = 99$ yields

$$t_{0.005} = 2.617, \ \textit{for } v = 120.$$

   The sample mean and standard deviation are given as

$$\bar{x} = 23500, \quad s = 3900.$$

   Thus the 99% confidence interval is

$$23500 - (2.617)\frac{3900}{\sqrt{100}} < \mu < 23500 + (2.617)\frac{3900}{\sqrt{100}}$$

$$22479 < \mu < 24520.$$

   (b) What can we assert with 99% confidence about the possible size of our error if we estimate the average number of kilometers driven by car owners in Virginia to be 23,500 kilometers per year?

   Solution: *Using our confidence interval above, and subtracting the sample mean from the left hand side, we can assert that with 99% confidence that 23,500 km per year is correct plus or minus*

$$23500 - 22479 = 1021 \ km.$$

   I.e., our true mean, with 99% confidence is

$$\mu = 23500 \pm 1021 \ km.$$

3. A machine produces metal pieces that are cylindrical in shape. A sample of these pieces is taken and the diameters are found to be 1.01, 0.97, 1.03, 1.04, 0.99, 0.98, 0.99, 1.01, and 1.03 centimeters.

(a) Find the sample mean and sample variance.

Solution: *Note that this is the same data as from 9.11 done earlier. So we have the calculated sample mean and variance are*

$$\bar{x} = 1.0056, \quad s^2 = 0.00060278 \implies s = \sqrt{s^2} = 0.0246.$$

(b) If the true standard deviation is $\sigma = 0.025$, use the Chi-squared distribution and the sample standard deviation to support or contest σ.

Solution: *We can develop a test by choosing, arbitrarily the middle 90% of the chi-squared distribution. That is, if the sample chi-squared value falls between* $\chi^2_{0.95}$ *and* $\chi^2_{0.05}$ *we support the* σ *value. Otherwise we dispute. Using the chi-squared table for* $v = 9-1 = 8$, *we find*

$$\chi^2_{0.95} = 2.733, \text{ and } \chi^2_{0.05} = 15.507 \text{ for } v = 8.$$

The sample chi-squared value is

$$\chi^2_s = \frac{(n-1)}{\sigma^2}s^2 = \frac{8}{0.025^2}0.0246^2 = 7.746.$$

This number satisfies

$$2.733 < 7.746 < 15.507$$

so the claim is supported.

(c) Using $\sigma = 0.025$, find a 99% confidence interval on the mean diameter.

Solution: *Since the population value of* $\sigma = 0.025$ *is known, we want to use the confidence interval for* σ *known. Thus the* $1-\alpha$ *confidence interval is*

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

where $z_{\alpha/2}$ *is the value from the Standard Normal distribution table for*

$$P(Z > z_{\alpha/2}) = \alpha/2.$$

For a 99% confidence interval we have $\alpha = 0.01$ *and* $\alpha/2 = 0.005$. *Using the table, we find a number close to 0.005 in the table and thus,*

$$z_{0.005} = -2.575.$$

We can drop the negative because symmetry gives

$$P(Z < -2.575) = P(Z > 2.575) \approx 0.0050.$$

Thus $z_{0.005} = 2.575$. *Thus the 95% confidence interval is*

$$1.0056 - (2.575)\frac{0.025}{\sqrt{9}} < \mu < 1.0056 + (2.575)\frac{0.025}{\sqrt{9}}$$

$$0.9841 < \mu < 1.0271.$$

(d) Using the only the sample standard deviation, find a 99% confidence interval on the mean diameter.

Solution: *Note, this problem was done in 9.11. Sorry for the repeat. It is copied below. Since the population value of σ is unknown, we want to use the confidence interval for σ unknown. Thus the $1-\alpha$ confidence interval is*

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ is the value from the t-distribution table for

$$P(T > t_{\alpha/2}) = \alpha/2.$$

For a 99% confidence interval we have $\alpha = 0.01$ and $\alpha/2 = 0.005$. Using the table for $v = 9-1 = 8$ yields

$$t_{0.005} = 3.355, \text{ for } v = 8.$$

The calculated sample mean and variance are

$$\bar{x} = 1.0056, \quad s^2 = 0.00060278 \implies s = \sqrt{s^2} = 0.0246.$$

Thus the 99% confidence interval is

$$1.0056 - (3.355)\frac{0.0246}{\sqrt{9}} < \mu < 1.0056 + (3.355)\frac{0.0246}{\sqrt{9}}$$

$$0.9781 < \mu < 1.0331.$$