

An Algebraic Exploration of Dominating Sets and Vizing's Conjecture

S. Margulies

Department of Mathematics
Pennsylvania State University
State College, PA

margulies@math.psu.edu

I. V. Hicks

Computational and Applied Mathematics
Rice University
Houston, TX

ivhicks@rice.edu

October 7, 2011

Abstract

Systems of polynomial equations are commonly used to model combinatorial problems such as independent set, graph coloring, Hamiltonian path, etc.. We formulate the dominating set problem as a system of polynomial equations in two different ways: first, as a single, high-degree polynomial, and second as a collection of polynomials based on the complements of domination-critical graphs. We then provide a sufficient criterion for demonstrating that a particular ideal representation is already the universal Gröbner bases of the ideal, and show that the second representation of the dominating set ideal in terms of domination-critical graphs is the universal Gröbner basis for that ideal. We then present the first algebraic formulation of Vizing's conjecture, and discuss the theoretical and computational ramifications to Vizing's conjecture of using either of the two dominating set representations described above.

1 Introduction

The combination of non-linear models and techniques from computer algebra is gaining acceptance as a tool for exposing combinatorial properties of graph-theoretic problems. For example, prior work on polynomial encodings includes colorings [3, 18, 10, 15, 22, 24, 25, 27, 20], stable sets [18, 17, 22, 28, 21], matchings [11], and flows [3, 27, 26]. These non-linear encodings have been used to prove combinatorial results ([2, 21, 19] and references therein), and also for computation [20], but techniques from computer algebra are certainly not widely used by combinatorists and graph theorists, and thus have not yet been deeply explored. In this paper, we move beyond simply formulating a graph-theoretic problem as a system of polynomial equations, and instead formulate an entire graph-theoretic conjecture using systems of polynomial equations and algebraic techniques. Although the method we introduce here can probably be applied to other open questions involving inequalities, we focus on the dominating set problem and an algebraic approach to a famous open question from domination theory: Vizing's conjecture.

Given a graph G , a set $D \subseteq V(G)$ is a *dominating set* if for all $v \notin D$, there is a $u \in D$ such that v is adjacent to u . The domination number of G , denoted by $\gamma(G)$, is the size of a minimum dominating set in G . The decision problem of determining whether a given graph has a dominating set of size k is NP-complete [12]. Given graphs G and H , the Cartesian product graph $G \square H$ has

vertex set $V(G) \times V(H)$, and edge set

$$E(G \square H) = \{(gh, g'h') : g = g' \text{ and } (h, h') \in E(H), \text{ or } h = h' \text{ and } (g, g') \in E(G)\},$$

where $g, g' \in V(G)$ and $h, h' \in V(H)$. In 1968, V. Vizing conjectured a beautiful relationship between domination numbers and Cartesian product graphs:

Conjecture 1.1 (Vizing [31], 1968) *Given graphs G and H , $\gamma(G)\gamma(H) \leq \gamma(G \square H)$.*

Vizing’s conjecture is an active area of research spanning over forty years. Early results have focused on proving the conjecture holds for a certain classes of graphs. For example, in 1979, Barcalkin and German [5] proved that Vizing’s conjecture holds for graphs satisfying a certain “partitioning condition” on the vertex set. The idea of a “partitioning condition” inspired work for the next several decades, as Vizing’s conjecture was shown to hold on paths, trees, cycles, chordal graphs, graphs satisfying certain coloring properties, and graphs with $\gamma(G) \leq 2$. These results are clearly outlined in the 1998 survey paper by Hartnell and Rall [14]. In 2000, Clark and Suen [7] showed that $\gamma(G)\gamma(H) \leq 2\gamma(G \square H)$, and in 2004, L. Sun [30] showed that Vizing’s conjecture holds on graphs with $\gamma(G) \leq 3$. Finally, in 2009, Hartnell and Rall, et al. [6] contributed another thorough survey paper summarizing the work from 1968 to 2008, which contains new results, new proofs of existing results, and comments about minimal counter-examples.

We begin by reviewing basic ideas from algebraic geometry: unions of varieties, intersections of ideals, notions of radical ideals, and universal Gröbner bases (Sections 2 and 3). In Section 3, for certain ideals, we develop a criterion for identifying a particular basis of the ideal as the universal Gröbner basis. In Section 4.1, we represent both the problem of finding graphs G with dominating sets of size k , and the problem of finding graphs G and H with dominating sets of size k, l respectively such that the Cartesian product graph $G \square H$ has a dominating set of size r , as systems of polynomial equations. In Section 4.2, we develop the idea of k -domination-covers, or k -covers, which are the complements of k -dominating graphs. We identify specific properties of k -covers, which thus translates to new results pertaining to domination-critical graphs, and briefly discuss known results in the active field of domination-critical graphs.

In Section 4.3, we unify the disparate results that have appeared thus far in our paper: we prove that the *same* ideals described in Section 4.1 can be represented by a set of polynomials based on k -covers, that is, by a set of polynomials based on the complements of domination-critical graphs. By the results in Section 3, we prove that this representation is the universal Gröbner basis of the ideal.

Our paper culminates in Section 5 with an algebraic representation of Vizing’s conjecture. This representation is built upon the union of certain varieties and the intersection of certain ideals. We initially present the algebraic version of Vizing’s conjecture without respect to a particular representation. We then discuss the consequences of using either of the two representations presented in Sections 4.1 and 4.3. We include comments about computational results and future computational directions, as well as approaches from a purely graph theory perspective. We conclude by clarifying the relationship between universal Gröbner bases and Vizing’s conjecture via k -covers, and reclaim a known result where Vizing’s conjecture holds.

2 Algebraic Definitions and Background

In this section, we outline the basic definitions and results concerning ideals and varieties that are utilized throughout the paper, and that are particularly necessary for the algebraic approach to

Vizing's conjecture described in Section 5. The results presented in this section are well-known in the field of algebraic geometry and are presented in detail in [8], Chapters 2 and 4. The only new contribution is Lemma 2.2, which is a small application of well-known results. The ideals referenced throughout are always ideals in polynomial rings, i.e. $I \subseteq \mathbb{K}[x_1, \dots, x_n]$, where \mathbb{K} is an algebraically-closed field. In our case, $\mathbb{K} = \mathbb{C}$.

Given a system of polynomial equations, $f_1 = f_2 = \dots = f_s = 0$, the ideal associated with the system is $I = \langle f_1, \dots, f_s \rangle$, and the *variety* associated I (i.e. $V(I)$) is the set of common zeros of $\{f_1, \dots, f_s\}$. In other words, $V(I) = \{\mathbf{x} \in \mathbb{C}^n : f_1(\mathbf{x}) = \dots = f_s(\mathbf{x}) = 0\}$. An ideal is *zero-dimensional* if $V(I)$ is finite. Throughout the paper, we will often write a polynomial $f \in I$ as $\sum(\cdot)f_i$. In this case, (\cdot) represents the coefficients of the generators f_i , but because we do not refer to the coefficients explicitly, we do not need to give them individual and precise labels such as a_i .

Given two ideals, $I = \langle f_1, \dots, f_s \rangle$ and $J = \langle g_1, \dots, g_t \rangle$, the *product ideal* $I \cdot J$ is the ideal generated by all polynomials $f \cdot g$ with $f \in I$ and $g \in J$. It can be shown ([8], pg 183, Prop. 6) that $I \cdot J$ (often denoted by IJ) is the ideal generated by $\langle f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t \rangle$. We summarize the results concerning the varieties of I and J as follows:

$$V(I) \cup V(J) = V(\langle f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t \rangle) = V(I \cdot J) = V(I \cap J) .$$

An ideal I is *radical* if $f^m \in I$, for some integer $m \geq 1$, implies that $f \in I$. Given an ideal I , the *radical of I* , denoted \sqrt{I} , is the set $\{f : f^m \in I \text{ for some integer } m \geq 1\}$. It is easy to see that an ideal I is radical if and only if $I = \sqrt{I}$. We recall the following fact concerning product ideals and radical ideals (left as an exercise in [8]): Given radical ideals $I, J \in \mathbb{K}[x_1, \dots, x_n]$, $\sqrt{IJ} = I \cap J$. We note that in one particular case, it is trivial to determine whether or not an ideal is radical.

Lemma 2.1 ([16], Section 3.7.B, pg. 246) *Given a zero-dimensional ideal I , if I contains a univariate square-free polynomial in each variable, then I is radical.*

In this case, *square-free* implies that when a polynomial is decomposed into its unique factorization, there are no repeated factors. For example, $(x^2 + y)(x^4 + 2z + 3)$ is square-free, but $(x^2 + y)(x^4 + 2z + 3)^3$ is not. In particular, Lemma 2.1 implies that ideals containing $x_i^2 - x_i = x_i(x_i - 1)$ in each variable (i.e., the boolean ideals) are radical. In Section 4, we will see that all ideals associated with dominating sets (and therefore Vizing's conjecture) are radical for this reason.

In general, it may be quite difficult to determine a basis for \sqrt{IJ} . However, if I and J are radical ideals via Lemma 2.1, and additionally, if the univariate, square-free polynomials in each variable contained in I, J are *identical*, we can explicitly determine a basis for \sqrt{IJ} .

Lemma 2.2 *Let I and J be ideals such that $I = \langle f_1, \dots, f_s \rangle$ and $J = \langle g_1, \dots, g_t \rangle$. Furthermore, for $1 \leq i \leq n$, let $f_i = g_i$ be square-free univariate polynomials in x_i . Then $\sqrt{IJ} = \langle f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t \rangle + \langle f_i : 1 \leq i \leq n \rangle$.*

Proof: We prove the inclusion in both directions. For convenience, let $M := \langle f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t \rangle + \langle f_i : 1 \leq i \leq n \rangle$. First, we show that $\sqrt{IJ} \subseteq M$. Let $h \in \sqrt{IJ}$. Then, there exists an integer $m \geq 1$ such that $h^m \in IJ$. Thus, $h^m = \sum(\cdot)fg$. Thus, $h^m \in M$. But, by Lemma 2.1, the ideal M is radical since M contains a square-free univariate polynomial f_i in each variable x_i . Thus, $h^m \in M$ implies that $h \in M$.

Conversely, to show that $M \subseteq \sqrt{IJ}$, let $h \in M$. Then, $h = \sum(\cdot)fg + \sum(\cdot)f$. Consider

$$h^2 = \left(\sum(\cdot)fg + \sum(\cdot)f \right) \left(\sum(\cdot)fg + \sum(\cdot)f \right) .$$

Any term in the expanded h^2 can be viewed as $(\cdot)f_i g_j f_k g_l = (\cdot)f_i g_j$, or $(\cdot)f_i g_j f_k = (\cdot)f_i g_j$, or $(\cdot)f_i f_j = (\cdot)f_i g_j$, etc. Thus, any term in the expanded h^2 can be written as $(\cdot)fg$, with any extra multiplicities in f or g simply folded into the coefficient. Therefore,

$$h^2 = \sum (\cdot)fg ,$$

and there exists an integer $m = 2 \geq 1$ such that $h^m \in IJ$, which implies that $h \in \sqrt{IJ}$. \square

This brings us to the following critical fact: if I and J are boolean, radical ideals, then by Lemma 2.2, $\sqrt{IJ} = \langle f_i g_j : 1 \leq i \leq s, 1 \leq j \leq t \rangle + \langle x_i(x_i - 1) : 1 \leq i \leq n \rangle = I \cap J$.

In Section 4, when we represent the dominating set problem as a system of polynomial equations, the representations will be boolean, radical ideals as described above. Thus, the basis of their product ideals can be described via Lemma 2.2. This fact will be vital in Section 5 when we present an algebraic representation of Vizing's conjecture.

3 Universal Gröbner Bases and Products of Linear Factors

In this section, we provide a brief overview of the terminology (from [8], Chap. 2) pertaining to Gröbner bases, and build off the ideas of De Loera in [18] to show that a specific set of linear factor polynomials is a universal Gröbner basis. These "linear factor" ideals allow us to provide a combinatorial interpretation of the universal Gröbner basis of the dominating set ideal defined in Section 4, and will be further used in our algebraic exploration of Vizing's conjecture.

A *monomial order* \prec for the monomials in the polynomial ring $\mathbb{K}[x_1, \dots, x_n]$ is a well-ordering which is multiplicative, and for which the constant polynomial is the smallest. The *leading term* $\text{LT}(f)$ of a polynomial $f \in \mathbb{K}[x_1, \dots, x_n]$ is the largest monomial in f (and its corresponding coefficient) with respect to the monomial order \prec . Given an ideal I , a finite subset $\mathcal{G} = \{g_1, \dots, g_r\}$ of I is a *Gröbner basis* of I (with respect to \prec) if

$$\langle \text{LT}_{\prec}(g_1), \dots, \text{LT}_{\prec}(g_r) \rangle = \langle \text{LT}_{\prec}(I) \rangle = \langle \text{LT}_{\prec}(f) : f \in I \rangle .$$

A finite subset \mathcal{G} of I is a *universal Gröbner basis* of I if it is Gröbner basis of I with respect to *any* monomial order. Given an ideal I , the universal Gröbner basis of I is unique.

There is a specific criterion, sometimes referred to as *Buchberger's S-pair criterion*, for determining if a given subset of polynomials is a Gröbner basis of I . Before we state Buchberger's S-pair criterion, we fix a monomial order \prec and recall the following notation:

- \bar{f}^F denotes the remainder of division of the polynomial f by the ordered s -tuple $F = (f_1, \dots, f_s)$.
- x^γ is the *least common multiple* of the leading monomial $\text{LM}(f)$ and the leading monomial $\text{LM}(g)$, written as $x^\gamma = \text{LCM}(\text{LM}(f), \text{LM}(g))$.
- The *S-pair* of f and g is the combination:

$$S(f, g) = \frac{x^\gamma}{\text{LT}(f)} \cdot f - \frac{x^\gamma}{\text{LT}(g)} \cdot g .$$

We will now characterize Gröbner bases in terms of S-pairs.

Theorem 3.1 ([8], Chap. 2, Sec. 6) *Given an ideal I , a basis $\mathcal{G} = \{g_1, \dots, g_r\}$ of I is a Gröbner basis for I , if and only if, for all pairs $i \neq j$, $\overline{S(g_i, g_j)}^{\mathcal{G}} = 0$.*

We will now construct a set of polynomials, where each polynomial is a product of linear factors, and show that when the set satisfies a special condition called the *linear factor criterion*, the set is a universal Gröbner basis. Let $S = \{i_1, i_2, \dots, i_k\} \subseteq \{1, \dots, n\}$, and let

$$P(S) := (x_{i_1} - 1)(x_{i_2} - 1) \cdots (x_{i_k} - 1), \quad \text{and} \quad x(S) = x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Definition 3.2 *Let $\{S_1, \dots, S_D\}$ be a set such that each $S_i \subseteq \{1, \dots, n\}$ and $|S_i| = d_i$. We say that the set $\{S_1, \dots, S_D\}$ satisfies the linear factor criterion if, for any integers i, j, k , S_k is a not a proper subset of $S_i \setminus S_i \cap S_j$, and S_k is a not a proper subset of $S_j \setminus S_i \cap S_j$.*

Theorem 3.3 *Let $\{S_1, \dots, S_D\}$ with $S_i \subseteq \{1, \dots, n\}$ be a set that satisfies the linear factor criterion, and let $g_i = P(S_i)$. Then $\{g_1, \dots, g_D\}$ is a universal Gröbner basis of $\langle g_1, \dots, g_D \rangle$.*

Before we prove this theorem, we present an example.

Example 3.4 *Consider $\mathbb{C}[x_1, \dots, x_{12}]$, and let $S_1 = \{1, 2, 3, 5, 6\}$, $S_2 = \{1, 2, 3, 7, 8\}$, $S_3 = \{9, 10, 11, 12\}$, $S_4 = \{4, 8, 9\}$. Let $\mathcal{G} = \{g_1, g_2, g_3, g_4\}$ with $g_i = P(S_i)$.*

$$\begin{aligned} g_1 &:= (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_5 - 1)(x_6 - 1), \\ g_2 &:= (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_7 - 1)(x_8 - 1), \\ g_3 &:= (x_9 - 1)(x_{10} - 1)(x_{11} - 1)(x_{12} - 1), \\ g_4 &:= (x_4 - 1)(x_8 - 1)(x_9 - 1). \end{aligned}$$

Since $S_1 \cap S_2 = \{1, 2, 3\}$, $S_1 \setminus (S_1 \cap S_2) = \{5, 6\}$, and $S_2 \setminus (S_1 \cap S_2) = \{7, 8\}$. Neither S_3 nor S_4 is a proper subset of $S_1 \setminus (S_1 \cap S_2)$ or $S_2 \setminus (S_1 \cap S_2)$. This is true for all i, j pairs, thus $\{S_1, S_2, S_3, S_4\}$ satisfies the linear factor criterion. Additionally, note that $\text{LT}(g_i)$ is always $x(S_i)$, regardless of the specified monomial order. Thus, we see

$$\begin{aligned} S(g_2, g_4) &= \frac{x_1 x_2 x_3 x_4 x_7 x_8 x_9}{x_1 x_2 x_3 x_7 x_8} g_2 - \frac{x_1 x_2 x_3 x_4 x_7 x_8 x_9}{x_4 x_8 x_9} g_4 = x_4 x_9 g_2 - x_1 x_2 x_3 x_7 g_4 \\ &= (x_4 + x_9 - 1)g_2 - (x_1 x_2 x_3 + x_1 x_2 x_7 + x_1 x_3 x_7 + x_2 x_3 x_7 - x_1 x_2 - x_1 x_3 - x_1 x_7 - x_2 x_3 \\ &\quad - x_2 x_7 - x_3 x_7 + x_1 + x_2 + x_3 + x_7 - 1)g_4. \end{aligned}$$

Regardless of the monomial order, the leading terms of the coefficients of g_2 and g_4 are not divisible by the leading terms of g_1, g_2, g_3 or g_4 . Thus, $\overline{S(g_2, g_4)}^{\mathcal{G}} = 0$. Performing similar calculations for all other pairs can quickly show that $\{g_1, \dots, g_4\}$ is the universal Gröbner basis of $\langle g_1, \dots, g_4 \rangle$. \square

For simplicity of notation, let $S_{\text{only}}^i = S_i \setminus (S_i \cap S_j)$ and $S_{\text{only}}^j = S_j \setminus (S_i \cap S_j)$. Furthermore, let S_{\subseteq}^{ij} be a subset of $S_i \cap S_j$, S_{\subseteq}^i be a subset of S_{only}^i , and S_{\subseteq}^j be a subset of S_{only}^j . Note that

$$|S_{\text{only}}^j| + d_i = |S_{\text{only}}^i| + d_j. \quad (1)$$

Proof: We will show that the set of generators $\mathcal{G} = \{g_1, \dots, g_D\}$ satisfies Buchberger's S-pair criterion. Consider any two polynomials g_i, g_j . We must show $\overline{S(g_i, g_j)}^{\mathcal{G}} = 0$. We claim

$$S(g_i, g_j) = \frac{x(S_i \cup S_j)}{x(S_i)} g_i - \frac{x(S_i \cup S_j)}{x(S_j)} g_j = x(S_{\text{only}}^j) g_i - x(S_{\text{only}}^i) g_j \quad (2)$$

$$= \left(\text{LT}(P(S_{\text{only}}^j)) - P(S_{\text{only}}^j) \right) g_i - \left(\text{LT}(P(S_{\text{only}}^i)) - P(S_{\text{only}}^i) \right) g_j . \quad (3)$$

Before we prove the equality between lines 2 and 3, we note that, if true, we have already shown that $\overline{S(g_i, g_j)}^{\mathcal{G}} = 0$. This can be seen by noting that, since $\{S_1, \dots, S_D\}$ satisfies the linear factor criterion, given integers i, j, k , S_k is not a proper subset of S_{only}^i and S_k is not a proper subset of S_{only}^j . Additionally, regardless of the monomial order, the leading term of g_i is $x(S_i)$, and the leading term of $\left(\text{LT}(P(S_{\text{only}}^j)) - P(S_{\text{only}}^j) \right)$ is a *proper* subset of S_{only}^j (similarly, $\left(\text{LT}(P(S_{\text{only}}^i)) - P(S_{\text{only}}^i) \right)$ is a *proper* subset of S_{only}^i). Thus, $\text{LT}(g_k)$ does not divide either $\left(\text{LT}(P(S_{\text{only}}^j)) - P(S_{\text{only}}^j) \right)$ or $\left(\text{LT}(P(S_{\text{only}}^i)) - P(S_{\text{only}}^i) \right)$, and we can see that $\overline{S(g_i, g_j)}^{\mathcal{G}} = 0$.

In order to prove the equality between lines 2 and 3, we must show that every monomial in line 2 either cancels within line 2, or appears in line 3 with the same coefficient, and vice versa.

Let $x(M)$ be a monomial appearing in line 2. Either

$$M = S_{\text{only}}^j \cup S_{\subseteq}^{ij} \cup S_{\subseteq}^i \text{ only} , \quad \text{or} \quad M = S_{\text{only}}^i \cup S_{\subseteq}^{ij} \cup S_{\subseteq}^j \text{ only} .$$

We note that $x(M)$ appears in both $x(S_{\text{only}}^j)g_i$ and $x(S_{\text{only}}^i)g_j$ only when $S_{\subseteq}^i \text{ only} = S_{\text{only}}^i$ and $S_{\subseteq}^j \text{ only} = S_{\text{only}}^j$. In this case, the coefficient for $x(M)$ is $(-1)^C - (-1)^D$ where

$$C = d_i - (|S_{\subseteq}^{ij}| + |S_{\text{only}}^i|) , \quad \text{and} \quad D = d_j - (|S_{\subseteq}^{ij}| + |S_{\text{only}}^j|) .$$

This formula can be explained by noting that, in any polynomial that is the product of linear factors, the coefficient for a given monomial is $(-1)^C$ where C is the degree of the polynomial minus the degree of the given monomial. For example, consider $f := (x_1 - 1)(x_2 - 1)(x_3 - 1)(x_7 - 1)$. The coefficient for the monomial $x_1x_2x_3$ is $(-1)^{4-3} = -1$, and the coefficient for x_2x_7 is $(-1)^{4-2} = 1$.

Using the identity given in Eq. 1, we note that $d_i - |S_{\text{only}}^i| = d_j - |S_{\text{only}}^j|$. Thus, $C = D$, and the monomial cancels.

We will now consider the case where $M = S_{\text{only}}^j \cup S_{\subseteq}^{ij} \cup S_{\subseteq}^i \text{ only}$ and $S_{\subseteq}^i \text{ only} \neq S_{\text{only}}^i$. In this case, $x(M)$ appears in line 2 in the product $x(S_{\text{only}}^j)g_i$, and the coefficient for $x(M)$ is $(-1)^C$ where

$$C = d_i - (|S_{\subseteq}^{ij}| + |S_{\subseteq}^i \text{ only}|) .$$

In line 3, $x(M)$ appears only in the product $-(\text{LT}(P(S_{\text{only}}^i)) - P(S_{\text{only}}^i))g_j$, and the coefficient for $x(M)$ is $(-1)^{C'}$ where

$$C' = |S_{\text{only}}^i| - |S_{\subseteq}^i \text{ only}| + d_j - (|S_{\subseteq}^{ij}| + |S_{\text{only}}^j|) + 2 .$$

But using the identity from Eq. 1, and substituting for d_j , we see

$$\begin{aligned} C' &= |S_{\text{only}}^i| - |S_{\subseteq}^i \text{ only}| + (|S_{\text{only}}^j| + d_i - |S_{\text{only}}^i|) - (|S_{\subseteq}^{ij}| + |S_{\text{only}}^j|) + 2 \\ &= d_i - |S_{\subseteq}^i \text{ only}| - |S_{\subseteq}^{ij}| + 2 . \end{aligned}$$

Thus, C is the same parity as C' , which implies that the coefficient for $x(M)$ in line 2, and the coefficient for $x(M)$ in line 3 are the same.

The case where $M = S_{\text{only}}^i \cup S_{\subseteq}^{ij} \cup S_{\subseteq}^j \text{ only}$ and $S_{\subseteq}^j \text{ only} \neq S_{\text{only}}^j$ is the same. Thus, we have shown that every monomial in line 2 either cancels, or appears in line 3 with the same coefficient.

We must now show that every monomial in line 3 either cancels, or appears in line 2 with the same coefficient. Let $x(M)$ be a monomial appearing in line 3. Either

$$M = S_{\underline{C}}^j \text{ only} \cup S_{\underline{C}}^{ij} \cup S_{\underline{C}}^i \text{ only} , \quad \text{or} \quad M = S_{\overline{C}}^i \text{ only} \cup S_{\underline{C}}^{ij} \cup S_{\underline{C}}^j \text{ only} .$$

We note that in each case, M contains a *proper* subset of S_{only}^j or a *proper* subset of S_{only}^i . Additionally, the two cases when $S_{\underline{C}}^i \text{ only} = S_{\text{only}}^i$, or $S_{\underline{C}}^j \text{ only} = S_{\text{only}}^j$ have already been explicated above. Thus, in both cases, we have already shown that $x(M)$ appears in line 2 with the same coefficient.

We will now consider monomials of the form $x(M)$ where $M = S_{\overline{C}}^j \text{ only} \cup S_{\underline{C}}^{ij} \cup S_{\overline{C}}^i \text{ only}$, and show that these monomials cancel within line 3. The coefficient for $x(M)$ is $(-1)^C - (-1)^D$ where

$$C = |S_{\text{only}}^j| - |S_{\overline{C}}^j \text{ only}| + d_i - \left(|S_{\underline{C}}^{ij}| + |S_{\overline{C}}^i \text{ only}| \right) , \quad \text{and}$$

$$D = |S_{\text{only}}^i| - |S_{\overline{C}}^i \text{ only}| + d_j - \left(|S_{\underline{C}}^{ij}| + |S_{\overline{C}}^j \text{ only}| \right) + 2 .$$

As before, using the identify from Eq. 1 and substituting for d_j , we see C is the *same parity* as D . Thus, $(-1)^C - (-1)^D = 0$, and the monomial cancels.

Thus, we have shown that every monomial in line 2 either cancels within line 2, or appears in line 3 with the same coefficient, and vice versa. Additionally, we have not utilized the properties of any particular monomial order to do so. Thus, we have shown that $\{g_1, \dots, g_D\}$ is a universal Gröbner basis for $\langle g_1, \dots, g_D \rangle$. \square

The following corollary extends the theorem above to include “boolean” polynomials of the form $x_i^2 - x_i$. This is particularly interesting because boolean polynomials are a common ingredient in non-linear models of combinatorial problems.

Corollary 3.5 *Let $\{S_1, \dots, S_D\}$ with $S_i \subseteq \{1, \dots, n\}$ (and $|S_i| > 1$) be a set that satisfies the linear factor criterion. Let $g_i = P(S_i)$, and let $b_i = x_i^2 - x_i$. Then $\mathcal{G} = \{b_1, \dots, b_n\} \cup \{g_1, \dots, g_D\}$ is the universal Gröbner basis for $\langle b_1, \dots, b_n \rangle + \langle g_1, \dots, g_D \rangle$.*

We observe that $x_i - 1$ and $x_i^2 - x_i$ are redundant equations, which explains the extra condition $|S_i| > 1, \forall i$.

Proof: By Theorem 3.3, we have already seen that $\{g_1, \dots, g_D\}$ is a universal Gröbner basis. Therefore, in order to show that \mathcal{G} satisfies Buchberger’s criterion, it remains to show that $\overline{S(b_i, b_j)}^{\mathcal{G}} = \overline{S(b_i, g_j)}^{\mathcal{G}} = 0$, without relying on properties of a particular monomial order.

Because monomial orders are by definition well-orders, $\text{LT}(b_i)$ is always x_i^2 , and $\text{LT}(g_i)$ is always $x(S_i)$, regardless of the specific monomial order.

We will first show that $\overline{S(b_i, b_j)}^{\mathcal{G}} = 0$ for $i \neq j$. Note that

$$S(b_i, b_j) = \frac{x_i^2 x_j^2}{x_i^2} (x_i^2 - x_i) - \frac{x_i^2 x_j^2}{x_j^2} (x_j^2 - x_j) = x_i^2 x_j - x_i x_j^2 = x_j (x_i^2 - x_i) - x_i (x_j^2 - x_j) .$$

Since $|S_i| > 1$, the coefficients $-x_i$ and x_j are not divisible by $\text{LT}(g_k), \forall k$. Thus, $\overline{S(b_i, b_j)}^{\mathcal{G}} = 0$.

We will now show that $\overline{S(b_i, g_j)}^{\mathcal{G}} = 0$, for all i, j pairs.

• **Case 1:** Choose an i such that $i \notin S_j$, and write $g_j := x(S_j) + P_{\text{rest}}$. Note

$$S(b_i, g_j) = \frac{x_i^2 x(S_j)}{x_i^2} (x_i^2 - x_i) - \frac{x_i^2 x(S_j)}{x(S_j)} (x(S_j) + P_{\text{rest}}) = -x_i x(S_j) - x_i^2 (P_{\text{rest}})$$

$$= -x_i(x(S_j) + P_{\text{rest}}) - P_{\text{rest}}(x_i^2 - x_i) = -x_i g_j - P_{\text{rest}}(x_i^2 - x_i) .$$

By the linear factor criterion, $\text{LT}(P_{\text{rest}})$ is not divisible by $\text{LT}(g_k)$ for any k . Thus, $\overline{S(b_i, g_j)}^{\mathcal{G}} = 0$.

- **Case 2:** Choose an i such that $i \in S_j$, and write $g_j := x(S_j) - x(S_j \setminus i) + P_{\text{rest}}$. Since $i \in S_j$, this implies that $\text{LCM}(\text{LT}(b_i), \text{LT}(g_j)) = x_i x(S_j) = x_i^2 x(S_j \setminus i)$. Then

$$S(b_i, g_j) = \frac{x_i^2 x(S_j \setminus i)}{x_i^2} (x_i^2 - x_i) - \frac{x_i x(S_j)}{x(S_j)} (x(S_j) - x(S_j \setminus i) + P_{\text{rest}}) = -x_i P_{\text{rest}} \quad (4)$$

$$= (x(S_j \setminus i) - P(S_j \setminus i))(x_i^2 - x_i) , \quad (5)$$

and $\overline{S(b_i, g_j)}^{\mathcal{G}} = 0$ by the linear factor criterion. The equality of lines 4 and 5 can be seen as follows. The polynomial $x(S_j \setminus i) - P(S_j \setminus i)$ is the polynomial $P(S_j \setminus i)$ with the signs changed and the leading term removed. When multiplied by a positive x_i^2 and a negative x_i , every monomial that appears in that product also appears in the product $-x_i(P_{\text{rest}})$. To see this clearly, we will consider a monomial $x(M)$ where M is a *proper* subset of S_j (this is the only kind of monomial appearing in P_{rest} on line 4). There are two cases:

- **Case 2a:** $i \in M$. When $x(M)$ is multiplied by $-x_i$ (line 4), the sign of the leading coefficient changes, and the degree increases by one. In other words, $x_i x(M)$ appears in expanded product $-x_i(P_{\text{rest}})$ with leading coefficient $(-1) \cdot (-1)^{d_j - |M|}$. However, $x(M \setminus i)$ appears in $P(S_j \setminus i)$ with *the same parity sign* as $x(M)$ in P_{rest} (i.e., $(-1)^{(d_j - 1) - (|M| - 1)}$). Thus, the product $-P(S_j \setminus i)(x_i^2)$ produces the same monomial with the same leading coefficient as $-x_i(P_{\text{rest}})$ (line 5) and equality between lines 4 and 5 is preserved.
- **Case 2b:** $i \notin M$. In this case, $x(M)$ appears in $P(S_j \setminus i)$, but with *opposite sign* as $x(M)$ in P_{rest} . In particular, $x(M)$ in P_{rest} has coefficient $(-1)^{d_j - |M|}$ as before, but $x(M)$ in $P(S_j \setminus i)$ has coefficient $(-1)^{(d_j - 1) - |M|}$. Thus, the product $-P(S_j \setminus i)(-x_i)$ produces the same monomial with the same leading coefficient as $-x_i(P_{\text{rest}})$ and equality between lines 4 and 5 is preserved.

In cases 2a and 2b, we demonstrated that the monomials produced by $-P(S_j \setminus i)(x_i^2)$ and $-P(S_j \setminus i)(-x_i)$ respectively, appear in line 4. Thus, we have accounted for every monomial in line 5, and we have shown that line 4 is equal to line 5.

We have shown that $\overline{S(b_i, b_j)}^{\mathcal{G}} = 0$ for $i \neq j$, and that $\overline{S(b_i, g_j)}^{\mathcal{G}} = 0$, for all i, j pairs, without relying on any properties of a monomial order. Since we have already shown that $\{g_1, \dots, g_D\}$ is a universal Gröbner basis, this means that we have shown that $\{b_1, \dots, b_n\} \cup \{g_1, \dots, g_D\}$ is a universal Gröbner basis of $\langle b_1, \dots, b_n \rangle + \langle g_1, \dots, g_D \rangle$. This concludes our proof. \square

In Section 4.3, we will represent the dominating set problem as a system of polynomial equations in such a way that the representation is already a universal basis. This representation follows the work of De Loera [18], where the graph coloring problem is also represented in such a way that it is a universal Gröbner basis. These kinds of representations may prove useful in the context of combinatorial ideal membership questions, and also with the advance of algorithms specifically tailored for finding universal Gröbner basis, such as [4].

4 Dominating Sets, Ideals and Gröbner Bases

In this section, we begin by formulating the dominating set problem as a system of polynomial equations. We extend this formulation to include a dominating set in a graph G , a dominating set in a graph H , and a dominating set in the product graph $G \square H$. We refer to this formulation (linking graphs G , H , and $G \square H$) in Section 5 during the algebraic formulation of Vizing's conjecture. In Section 4.2, we introduce the idea of a k -domination cover or a k -cover. We explain the relation between k -covers and domination-critical graphs, and provide various examples of k -covers. We also prove several properties of k -covers, and provide a conjecture for future work. We unify these ideas in Section 4.3 by showing that k -covers provide the combinatorial interpretation of the universal Gröbner basis of the ideals described in Section 4.1. Thus, the purpose of this section is to explore two different non-linear models of the dominating set problem based on two different combinatorial properties, and surprisingly, the second representation is the universal Gröbner basis of the first.

4.1 Dominating Sets, Product Graphs and Systems of Polynomial Equations

Let S_k^n represent the set of k -subsets of $\{1, 2, \dots, n\}$. Thus, $|S_k^n| = \binom{n}{k}$, and $S \in S_k^n$ implies that $S \subseteq \{1, 2, \dots, n\}$ is a particular subset with $|S| = k$. In the following system of polynomial equations, there is one variable for every possible edge in a graph with n vertices. Thus, the variable e_{ij} refers to the edge between vertex i and vertex j . Since our graphs are undirected, we implicitly assume the substitution $e_{ji} = e_{ij}$ whenever $j > i$.

Theorem 4.1 *There is a bijection between the set of solutions of the following system of polynomial equations and the labeled graphs G in n vertices with a dominating set of size k .*

$$\begin{aligned} e_{ij}^2 - e_{ij} &= 0, \quad \text{for } 1 \leq i < j \leq n, \\ \prod_{S \in S_k^n} \left(\sum_{i \notin S} \left(\prod_{j \in S} (e_{ij} - 1) \right) \right) &= 0. \end{aligned} \quad (6)$$

Example 4.2 *Let $n = 3$ and $k = 1$. The variables in the system of polynomial equations defined by Theorem 4.1 are e_{12}, e_{13} and e_{23} . Furthermore, $S_1^3 = \{\{1\}, \{2\}, \{3\}\}$.*

For the second equation, we have:

$$((e_{21} - 1) + (e_{31} - 1))((e_{12} - 1) + (e_{32} - 1))((e_{13} - 1) + (e_{23} - 1)) = 0.$$

However, as noted above, since the graph is undirected, we implicitly assume the substitution $e_{21} = e_{12}$, $e_{31} = e_{13}$ and $e_{32} = e_{23}$. Thus, the system of equations is as follows:

$$\begin{aligned} e_{12}^2 - e_{12} &= 0, & e_{13}^2 - e_{13} &= 0, & e_{23}^2 - e_{23} &= 0 \\ ((e_{12} - 1) + (e_{13} - 1)) &((e_{12} - 1) + (e_{23} - 1)) &((e_{13} - 1) + (e_{23} - 1)) &= 0. \end{aligned}$$

□

Proof: We refer to the system of polynomial equations defined by Theorem 4.1 as \star throughout this proof. We will define a map ϕ between the solutions of \star and the labeled graphs G in n vertices with a dominating set of size k , and show that ϕ is a bijection between the two sets.

To deal with the problem of graph isomorphisms, we consider a set of n vertices with a fixed labeling 1 through n . Thus, two graphs G and G' on this labeled set of n vertices are equal if

and only if $(i, j) \in E(G)$ implies that $(i, j) \in E(G')$. In other words, if G and G' are isomorphic, they are not necessarily equal under this definition. Then, our map ϕ simply takes a solution to \star , and converts it to a graph G on the labeled set of n vertices by adding the edges (i, j) to $E(G)$ if and only if the variable $e_{ij} = 1$. Then, ϕ is one-to-one, since given any two solutions s and s' , if $G = \phi(s) = \phi(s') = G'$, then clearly $s = s'$.

We will now show that the image of ϕ is a subset of the set of labeled graphs G with a dominating set of size k . The boolean equations $e_{ij}^2 - e_{ij} = 0$ force every variable e_{ij} to be zero or one; thus, the boolean equations turn edges “on” or “off” when applying the map ϕ , and every solution corresponds to a particular graph G . We must now show that since the solution satisfies Eq. 6, the particular graph G formed from the solution has a dominating set of size k . Let A and B denote different pieces of Eq. 6 as follows.

$$\prod_{S \in S_k^n} \left(\underbrace{\sum_{i \notin S} \left(\underbrace{\prod_{j \in S} (e_{ij} - 1)}_A \right)}_B \right) = 0 .$$

The equation is only satisfied if one of the inner summations B (corresponding to some set S) is zero. The value of an individual summand A in B is either zero or ± 1 . However, two different summands A, A' in the summation B can never mutually cancel, since if they are both non-zero, they are both $-1^{|S|}$. Thus, they are either both -1 or both $+1$. Therefore, a summation B is only zero if every individual summand A is also zero, and A is only zero if at least one edge variable has value one. Therefore, every $i \notin S$ is adjacent to a $j \in S$. In other words, G has a dominating set of size k .

Having shown that the image of ϕ is a subset of the set of labeled graphs G with a dominating set of size k , we must now show that ϕ is onto, or that given any graph G with a dominating set of size k , $\phi^{-1}(G)$ is a solution to \star . The map $\phi^{-1}(G)$ would be applied as follows: if the edge e_{ij} is present in G , turn it “on” by setting the variable $e_{ij} = 1$. If the edge e_{ij} is not present in G , turn the variable e_{ij} “off” by setting the variable $e_{ij} = 0$. Clearly, the boolean equations $e_{ij}^2 - e_{ij} = 0$ are satisfied. Since the graph G has a dominating set of size k , let $S = \{i_1, \dots, i_k\}$ be such a dominating set. Thus, every vertex i not in the dominating set S must be adjacent to a vertex j that is in the dominating set S . In other words, an edge e_{ij} is “on” from $i \notin S$ to $j \in S$, and the $(e_{ij} - 1)$ term in the product corresponding to the dominating set S is equal to zero. Since every vertex $i \notin S$ must satisfy this condition, this implies that every summand in the summation (corresponding to the dominating set S)

$$\sum_{i \notin S} \left(\prod_{j \in S} (e_{ij} - 1) \right)$$

is equal to zero, and thus the entire summation is equal to zero. Since Eq. 6 is a product of summations, the equation is satisfied.

Thus, ϕ is one-to-one and onto, and ϕ is a bijection between solutions of \star and the set of labeled graphs G with a dominating set of size k . \square

We will now use the system of polynomial equations defined in Theorem 4.1 as a building block to model dominating sets in G, H and $G \square H$.

Theorem 4.3 *There is a bijection between the set of solutions of the following system of polynomial equations and the set of labeled graphs G, H in n, n' vertices with dominating sets of size k, l*

respectively such that their Cartesian product graph $G \square H$ has a dominating set of size r .

Representing a graph G in n vertices with a dominating set of size k :

$$e_{ij}^2 - e_{ij} = 0, \quad \text{for } 1 \leq i < j \leq n,$$

$$\underbrace{\prod_{S \in S_k^n} \left(\sum_{i \notin S} \left(\prod_{j \in S} (e_{ij} - 1) \right) \right)}_{P_G^k} = 0.$$

Representing the graph H in n' vertices with a dominating set of size l :

$$e'_{ij}{}^2 - e'_{ij} = 0, \quad \text{for } 1 \leq i < j \leq n',$$

$$\underbrace{\prod_{S \in S_l^{n'}} \left(\sum_{i \notin S} \left(\prod_{j \in S} (e'_{ij} - 1) \right) \right)}_{P_H^l} = 0.$$

Representing the Cartesian product graph $G \square H$ with a dominating set of size r :

$$\underbrace{\prod_{S \in S_r^{nn'}} \left(\sum_{gh \notin S} \left(\prod_{g'h \in S} (e_{gg'} - 1) \prod_{gh' \in S} (e'_{hh'} - 1) \right) \right)}_{P_{G \square H}^r} = 0.$$

Proof: It is clear from Theorem 4.1 that there is a bijection between the solutions of the equations representing graphs G and H and the set of graphs in n, n' vertices with dominating sets of size k, l respectively. The equation representing $G \square H$ is of the same form, except that this equation takes into account the unique structure of the product graph. For example, if a vertex not in S is adjacent to a vertex in S , the adjacency is either due to an edge in G , or an edge in H . In particular, $gh \notin S$ is either adjacent to a vertex $g'h \in S$ if the edge (g, g') is “on” in G , or $gh \notin S$ is adjacent to a vertex $gh' \in S$ if the edge (h, h') is “on” in H . Thus, the proof of the bijection follows the logic of the proof of Theorem 4.1. \square

Since the system of polynomial equations described in Theorem 4.3 depends on the variables n, k, n', l and r , we define the ideal $I(n, k, n', l, r)$ as

$$I(n, k, n', l, r) := \langle P_G^k, P_H^l, P_{G \square H}^r, e^2 - e \rangle,$$

where $e^2 - e$ denotes the entire set of boolean edge equations

$$\{e_{ij}^2 - e_{ij} : 1 \leq i < j \leq n, e'_{ij}{}^2 - e'_{ij} : 1 \leq i < j \leq n', \}.$$

We note that $I(n, k, n', l, r)$ is radical by Lemma 2.1, since the boolean equations $e^2 - e$ are univariate and square-free. We now define k -covers, which provide a combinatorial interpretation of the universal Gröbner basis of $I(n, k, n', l, r)$ in Section 4.3. We note that what is important about the next section is not the uniqueness or complexity of the definition of k -covers, but rather the interesting relationship between the ideal $I(n, k, n', l, r)$, the combinatorial idea of a k -cover, and the universal Gröbner basis of the ideal.

4.2 k -Covers and k -Dominating Sets

We begin by recalling that a graph G is *domination-critical* if, for any two non-adjacent vertices u, v , the graph $G' := G + (u, v)$ has $\gamma(G') = \gamma(G) - 1$. In other words, a graph G is domination-critical if, whenever any edge is added to G , the domination number decreases by one. Thus, k -domination-critical graphs are created from k -dominating graphs by adding edges. In this section, we study the complements of domination-critical graphs. We define a k -domination-cover (or k -cover) as analogous to a k -dominating graph, and a minimal k -cover as analogous to a k -domination-critical graph. Just as a domination-critical graph is obtained from a k -dominating graph by adding edges, a minimal k -cover is obtained from a k -cover by *removing* edges.

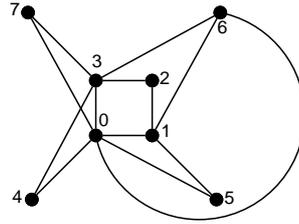
Before we begin, we recall a few definitions and introduce some notation. K_n denotes the complete graph in n vertices (an n -clique). For $S \subseteq V(G)$, the *open neighborhood* of S (denoted by $N_G(S)$, or simply $N(S)$ when the context of the graph is clear) is defined to be

$$N_G(S) := \{v : (u, v) \in E(G) \text{ and } u \in S \text{ and } v \notin S\}.$$

The *common neighborhood* of $S \subseteq V(G)$ (denoted by $cmN_G(S)$, or simply $cmN(S)$ when the context of the graph is clear) is defined to be

$$cmN_G(S) := \bigcap_{u \in S} N_G(u).$$

Example 4.4 *In the following example, let $S = \{0, 1, 3\}$. Then $cmN(S) = \{6\}$.*

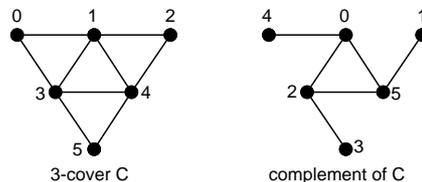


Definition 4.5 *A graph C is a k -cover if $cmN_C(S) \neq \emptyset$ for all $S \subseteq V(C)$ with $|S| = k - 1$.*

We say that a graph C is a k -cover if every set of size $k - 1$ has a common neighbor. We note that k -covers are only defined for $k \geq 2$.

Definition 4.6 *A k -cover C is minimal if for all $e \in E(C)$, $C' := C - e$ is not a k -cover.*

Example 4.7 *Here we see a minimal 3-cover (left) and its complement (right).*



We refer to graphs of the type described by Definition 4.5 as k -covers, because we are taking n vertices and then “covering” those n vertices with k -cliques such that the k -cliques intersect in a very particular way. Notice that the 3-cover illustrated in Ex. 4.7 takes six vertices, and then “covers” those six vertices with triangles such that any subset of size two has a common neighbor.

Proposition 4.8 *Given a k -cover C , every $v \in V(C)$ appears in a clique of size k .*

Proof: Since every set of size $k - 1$ in $V(C)$ has a common neighbor, every vertex has at least one outgoing edge. Thus, for any $v_1 \in V(C)$, v_1 is adjacent to some v_2 . If $2 \leq k - 1$, both v_1 and v_2 are each adjacent to a third vertex v_3 . In other words, v_1, v_2 and v_3 form a 3-clique. By repeating this process $k - 1$ times, we form a k -clique containing v_1 . Since v_1 was an arbitrary starting point, this algorithm can be repeated for any vertex, and every $v \in V(C)$ appears in a clique of size k . \square

We also observe that while 2-covers can be disconnected, k -covers with $k \geq 3$ are connected. Additionally, since the diameter of a graph is the longest shortest path between any two vertices, and every two vertices has a common neighbor, k -covers with $k \geq 3$ have diameter at most two.

Proposition 4.9 *A graph G is k -domination-critical if and only if \overline{G} is a minimal k -cover.*

Proof: If G is k -domination-critical, then G has a dominating set of size k , but no dominating set of size $k - 1$. Thus, every $(k - 1)$ -subset of vertices in \overline{G} has a common neighbor, and \overline{G} is a k -cover. Furthermore, since G is k -domination-critical, for any two non-adjacent vertices u, v , $\gamma(G + (u, v)) = k - 1$. In other words, if any edge (u, v) is removed from \overline{G} , then there is at least one $(k - 1)$ -subset of vertices that no longer has a common neighbor. Thus, \overline{G} is a minimal k -cover.

Conversely, assume that \overline{G} is a minimal k -cover. Then, for all $e \in E(\overline{G})$, $\overline{G} - e$ is *not* a k -cover. This implies that there is some set $D \subseteq V(\overline{G})$ of size $k - 1$ that does *not* have a common neighbor. In other words, for all $v \in (V(\overline{G}) \setminus D)$, v is *not* adjacent to some vertex in D . In other words, D is a dominating set of size $k - 1$ in G . Thus, we have shown that when any edge is *added* to G (or *removed* from \overline{G}), there is a dominating set of size $k - 1$ in G . Thus, in order to prove that G is k -domination-critical, it remains to show that there exists a dominating set of size k in G , and that there does *not* exist a dominating set of size $k - 1$ in G .

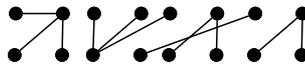
Clearly, there is no dominating set of size $k - 1$ in G , since \overline{G} is a k -cover. We will now show that G contains a dominating set of size k . Consider any two non-adjacent vertices $u, v \in V(G)$. Without loss of generality, let $D = \{i_1, \dots, i_{k-2}, u\}$ be the dominating set of size $k - 1$ in $G + (u, v)$. Since D is a dominating set in $G + (u, v)$, but not a dominating set in G , the only vertex that D does *not* dominate in G is v . Thus, $D + v$ is a dominating set in G , and since $|D + v| = k$, G is k -domination-critical. \square

We note that in [29], Sumner and Blich extensively explore properties of 3-domination-critical graphs. They explicitly categorize 2-domination graphs with the following theorem:

Theorem 4.10 (Sumner and Blich, 1983) *A graph G is 2-domination-critical if and only if $\overline{G} = \cup_{i=1}^n K_{1,n_i}$ with $n \geq 1$.*

The Sumner-Blich categorization of 2-domination-critical graphs is equivalent to Definition 4.5, since for any 2-cover, every vertex is adjacent to at least one other vertex.

Example 4.11 *Here we see a 2-cover on 16 vertices. We note that this 2-cover is the union of two $K_{1,3}$ graphs, two $K_{1,2}$ graphs, and a $K_{1,1}$ graph. Additionally, every single vertex (every set S with $|S| = 1$) has a common neighbor.*



Our categorization is an extension of the Sumner-Blicht categorization since Definition 4.5 is generalized for the complements of k -dominating graphs, although further characterizations are needed for minimal k -covers. We now link minimal $k + 1$ -covers with k -dominating graphs.

Theorem 4.12 *A graph $G = (V, E)$ has a dominating set of size k if and only if for all minimal $k + 1$ -covers $C = (V, E_{cov})$, $E(G) \cap E_{cov} \neq \emptyset$.*

Via this theorem, we characterize graphs with dominating sets of size one in terms of 2-covers. Since one is the smallest dominating set, this explains why we only define k -covers for $k \geq 2$.

Proof: Assume $G = (V, E)$ has a dominating set $D \subseteq V(G)$ of size k . Consider any minimal $k + 1$ -cover $C = (V, E_{cov})$. Since D is also a subset of $V(C)$, and since $|D| = k$, by Definition 4.5, D has a common neighbor v in C . Thus, for every $u \in D$, $(u, v) \in E_{cov}$. Since D is a dominating set in G and $v \notin D$, there must exist an edge $(u, v) \in E(G)$ with $u \in D$. Since $(u, v) \in E(G)$ and $(u, v) \in E_{cov}$, $E(G) \cap E_{cov} \neq \emptyset$.

Conversely, assume that for all minimal $k + 1$ -covers $C = (V, E_{cov})$, $E(G) \cap E_{cov} \neq \emptyset$. We must show that G contains a dominating set of size k . We proceed by contradiction. Assume that G does *not* contain a dominating set of size k , and form a new graph G_{crit} by adding edges to G until the resulting graph is $k + 1$ -domination-critical. Thus, $E(G) \subseteq E(G_{crit})$, and G_{crit} has a dominating set of size $k + 1$, but no dominating set of size k .

Next, we consider the complement of G_{crit} (denoted by $\overline{G_{crit}}$). Since G_{crit} is $k + 1$ -critical, $\overline{G_{crit}}$ is a minimal $k + 1$ -cover by Proposition 4.9. We note that $E(\overline{G_{crit}}) \subseteq E(G)$, and $E(\overline{G_{crit}}) \cap E(G_{crit}) = \emptyset$. However, $E(G) \subseteq E(G_{crit})$. Therefore, $E(\overline{G_{crit}}) \cap E(G) = \emptyset$. But this is a contradiction, since $\overline{G_{crit}}$ is a minimal $k + 1$ -cover, and $E(G) \cap E_{cov} \neq \emptyset$, for all minimal $k + 1$ -covers $C = (V, E_{cov})$. Thus, G must contain a dominating set of size k . \square

We note that, according to Thm. 4.12, in order to demonstrate that a graph G does *not* have dominating set of size k , it is sufficient to produce a $k + 1$ -cover C that is a subgraph of \overline{G} . This $k + 1$ -cover then becomes a *certificate* of the *non-existence* of a k -dominating set: it is a *coNP certificate*. However, even though the cover itself is polynomial-size in $|G|$, there are no known polynomial-time algorithms for verifying that every k -subset of C has a common neighbor. Therefore, there is no conflict between the use of k -covers as certificates, and the conjectured inequality of NP and coNP.

We now explore properties of k -covers.

Theorem 4.13 *A graph C is a k -cover if and only if, for all $S \subseteq V(C)$ with $1 \leq |S| \leq k - 1$, there exists a clique Q in $cmN_C(S)$ such that $|Q| = k - |S|$. Moreover, every $v \in cmN_C(S)$ appears in a clique Q of size $k - |S|$.*

Proof: Assume that a graph C is a k -cover. Given $S \subseteq V(C)$, if $|S| = k - 1$, we know by definition of a k -cover that $cmN(S)$ is non-empty. Therefore, a clique Q of size one exists in $cmN(S)$, and moreover, all vertices in $cmN(S)$ trivially form cliques of size one.

Now consider $S \subseteq V(C)$ with $1 \leq |S| \leq k - 2$. As before, by definition, since $|S| < k - 1$, $cmN(S)$ is non-empty. Let $v_1 \in cmN(S)$. Since $|S \cup v_1| \leq k - 1$, the set $S \cup v_1$ has a common neighbor v_2 . Thus, the set $\{v_1, v_2\}$ forms a clique of size two in $cmN(S)$. If $|S \cup v_1 \cup v_2| \leq k - 1$, we repeat this operation, and we note that the set $S \cup v_1 \cup v_2$ has a common neighbor v_3 , and that $\{v_1, v_2, v_3\}$ forms a clique of size three in $cmN(S)$. We repeat this operation exactly $k - |S| - 1$ times until we have formed the $k - |S|$ -clique $Q = \{v_1, v_2, \dots, v_{k-|S|}\}$ in $cmN(S)$.

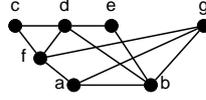
Therefore, for all $1 \leq |S| \leq k - 1$, there exists a clique Q in $cmN(S)$ such that $|Q| = k - |S|$. Moreover, every $v \in cmN(S)$ appears in a clique Q of size $k - |S|$, since this operation can be repeated with reference to any vertex v .

Conversely, let C be a graph such that for all $S \subseteq V(C)$ with $1 \leq |S| \leq k - 1$, there exists a clique Q in $cmN(S)$ such that $|Q| = k - |S|$. In particular, let S be a subset of $V(C)$ with $|S| = k - 1$. Then, there exists a clique Q of size one in $cmN(S)$. Therefore, by definition, C is a k -cover. \square

We have already shown that every vertex in a k -cover appears in a k -clique. However, we have not discussed the intersection of these k -cliques. We present the following conjecture.

Conjecture 4.14 *A graph C is a k -cover if and only if for all $S \subseteq V(C)$ with $|S| = k - 1$, there exists a map q from $v \in S$ to k -cliques in C such that for any $u, v \in S$, $|q(u) \cap q(v)| \geq k - 2$.*

Example 4.15 *Here is an example of Conjecture 4.14 on a 3-cover of 7 vertices.*



Consider $S = \{c, g\}$. Then, if $q(c) = \{c, d, f\}$ and $q(g) = \{a, f, g\}$, $|q(c) \cap q(g)| = |\{f\}| = 1 = k - 2$. Additionally, consider $S = \{e, g\}$. Then, if $q(e) = \{b, d, e\}$ and $q(g) = \{a, b, g\}$, $|q(e) \cap q(g)| = |\{b\}| = 1 = k - 2$. Upon inspection, we can see that given any set S with $|S| = k - 1 = 2$, a similar map q can be constructed and the conjecture holds. \square

4.3 k -Covers, k -Dominating Sets and a Universal Gröbner Basis of $I(n, k, n', l, r)$

In Section 4.1, we outlined a representation of the k -dominating set problem as a system of polynomial equations, and defined the ideal $I(n, k, n', l, r)$. In Theorem 4.1, $I(n, k, n', l, r)$ was defined by three, high-degree polynomials P_G^k, P_H^l and $P_{G \square H}^r$, and their associated boolean edge equations. Each of these polynomials was a series of products, with each product a sum of products. Thus, the polynomials P_G^k, P_H^l and $P_{G \square H}^r$ were a brute-force enumeration of every possible vertex set of size k , dominating or otherwise, in the graph.

In this section, we present another representation of the ideal $I(n, k, n', l, r)$. This representation is based on $k + 1$ -covers, with each polynomial equation corresponding to a minimal $k + 1$ -cover of $V(G)$. Thus, $I(n, k, n', l, r)$ can also be generated by many different polynomials of comparatively low degree, as opposed to only three polynomials of high degree. However, there is currently no known algorithm for enumerating minimal $k + 1$ -covers, which is equivalent to the problem of enumerating $k + 1$ -domination-critical graphs. Thus, another point of comparison between the representation expressed here and Theorem 4.1 is that the polynomials described in Theorem 4.1 can be explicitly written down, whereas polynomials based on $k + 1$ -covers, while combinatorially explicit, may be difficult to write down explicitly enough for the purpose of computation. However, the potential advantage of the cover representation is that $k + 1$ -covers are collections of edges that satisfy the linear factor criterion (Definition 3.2). Therefore, by Theorem 3.3 and Corollary 3.5, the representation described below is a universal Gröbner basis.

In presenting a combinatorial interpretation of any Gröbner basis, we must be particularly careful of input instances where the system of polynomial equations is *infeasible*. In these cases,

the associated variety is empty and the associated ideal is the entire ring. This implies that the universal Gröbner basis is simply the number one. In our case, we will exclude input instances where the system of polynomial equations is infeasible by restricting the values of n, n', k and l . However, in Theorem 5.11, we will show that the instances lost by this restriction correspond exactly to instances where Vizing's conjecture is known to be true.

Before we begin, we introduce some notation. Let \mathcal{C}_{k+1}^n be the set of all minimal $k+1$ -covers on n vertices. For $C \in \mathcal{C}_{k+1}^n$, let

$$P(C) := \prod_{e \in E(C)} (e - 1) .$$

Theorem 4.16 *There is a bijection between the set of solutions of the following system of polynomial equations and the labeled graphs G in n vertices with a dominating set of size k .*

$$\begin{aligned} e_{ij}^2 - e_{ij} &= 0 , & \text{for } 1 \leq i < j \leq n , \\ P(C) &= 0 , & \text{for each } C \in \mathcal{C}_{k+1}^n . \end{aligned} \tag{7}$$

We note that if $k = n$, there are no $n+1$ covers on n vertices. Thus, $\mathcal{C}_{n+1}^n = \emptyset$, and the only equations that appear are of the form $e^2 - e = 0$. Thus, any graph is a solution to this system, which is reasonable since every graph has a dominating set of size n .

Proof: As in the proof of Theorem 4.1, we must define a map ϕ and show that ϕ provides a bijection between the set of solutions and the set of graphs with a dominating set of size k . We will use the same map ϕ defined in the proof of Theorem 4.1, and thus, we already know that the map is one-to-one, and we only need show that the image of ϕ is the set of graphs with a dominating set of size k , and that the map is onto.

To consider the image of the map ϕ , we note that, as in Theorem 4.1, every solution corresponds to a particular graph G (the assignment of the variables e_{ij} to zero or one simply turns the edges “on” or “off” in G). Since the cover equations (Eqs. 7) are satisfied, this implies that for all minimal $k+1$ -covers, $E(G) \cap E(C) \neq \emptyset$. Then, by Theorem 4.12, G has a dominating set of size k .

To show that ϕ is onto, consider a graph G in n vertices that has a dominating set of size k . As before, we must show that $\phi^{-1}(G)$ (defined by setting variables e_{ij} to zero or one, depending on whether or not $e_{ij} \in E(G)$) maps to a solution. Clearly, the boolean equations $e_{ij}^2 - e_{ij} = 0$ are satisfied. Since the graph G has a dominating set of size k , for any minimal $k+1$ -cover C , $E(G) \cap E(C) \neq \emptyset$ by Theorem 4.12. Therefore, at least one edge in every cover is “on” and the cover equations (Eqs. 7) are satisfied. \square

Let $I(n, k)$ denote the ideal generated by polynomials described in Theorem 4.16. We will now prove that this representation is a universal Gröbner basis. We note that we discovered the connection between covers and Gröbner bases by experimental investigations using CoCoA Lib [1].

Corollary 4.17 *The basis $I(n, k)$ described in Theorem 4.16 is a universal Gröbner basis.*

Proof: We must show that $\{P(C) : C \in \mathcal{C}_{k+1}^n\}$ satisfies the linear factor criterion (Def. 3.2). Let C_i, C_j, C_k be covers in \mathcal{C}_{k+1}^n . Since each are minimal covers, it is clear that C_k can never be a proper subset of $C_i \setminus C_i \cap C_j$ or $C_i \setminus C_i \cap C_j$. From Theorem 3.3 and Corollary 3.5, we can see that the basis of $I(n, k)$ described in Theorem 4.16 is a universal Gröbner basis. \square

In Section 4.1, we used the system of polynomial equations defined in Theorem 4.1 to model the larger question of dominating sets in graphs G, H and $G \square H$ (Theorem 4.3). We will repeat the process here, using the cover-based model from Theorem 4.16 as a building block. However, a system of polynomial equations based on $k+1$ -covers is also intrinsically based on graph intersections. Thus, we must define the edge variables that appear in the intersection of an arbitrary r -cover on nn' vertices and an arbitrary product graph.

Definition 4.18 *Given graphs G and H , let C be a graph on the vertices $V(G) \times V(H)$. Then the set $\cap_{\square} C \subseteq (E(G) \cup E(H))$ is defined to be the following set of edges:*

$$\cap_{\square} C = \{(g, g') : (gh, g'h) \in E(C)\} \cup \{(h, h') : (gh, gh') \in E(C)\} .$$

Now we present the cover-based model that links dominating sets in G, H and $G \square H$. As before, this system of polynomial equations will be used in the algebraic representation of Vizing's conjecture in Section 5, and the restriction on the values of n, k, n', l and r will be more meaningful in that context.

Theorem 4.19 *Let n, k, n', l and r be integers such that $r \geq \min(n, n')$. There is a bijection between the set of solutions of the following system of polynomial equations and the set of labeled graphs G, H in n, n' vertices with dominating sets of size k, l respectively such that their Cartesian product graph $G \square H$ has a dominating set of size r .*

Representing a graph G in n vertices with a dominating set of size k :

$$\begin{aligned} e_{ij}^2 - e_{ij} &= 0, \quad \text{for } 1 \leq i < j \leq n, \\ P(C) &= 0, \quad \text{for each } C \in \mathcal{C}_{k+1}^n . \end{aligned}$$

Representing the graph H in n' vertices with a dominating set of size l :

$$\begin{aligned} e'_{ij}{}^2 - e'_{ij} &= 0, \quad \text{for } 1 \leq i < j \leq n', \\ P(C) &= 0, \quad \text{for each } C \in \mathcal{C}_{l+1}^{n'} . \end{aligned}$$

Representing the Cartesian product graph $G \square H$ with a dominating set of size r :

$$P(\cap_{\square} C) = 0, \quad \text{for each } C \in \mathcal{C}_{r+1}^{nn'} .$$

Proof: It is clear from Theorem 4.16 that there is a bijection between the solutions of the equations representing graphs G and H and the set of labeled graphs in n, n' vertices with dominating sets of size k, l respectively. The equation representing $G \square H$ is of the same form, except that this equation takes into account the unique structure of the product graph.

Assume that $G \square H$ has a dominating set $D \subseteq V(G \square H)$ of size r . We will show that the cover equations associated with $G \square H$ are satisfied. Let $C_{r+1}^{nn'}$ be a minimal $r+1$ -cover on $V(G \square H)$. Since $C_{r+1}^{nn'}$ is an $r+1$ -cover, every subset of size r has a common neighbor in $V(C_{r+1}^{nn'}) = V(G \square H)$. In particular, D has a common neighbor $gh \in V(C_{r+1}^{nn'})$. Since D is a dominating set of $G \square H$ and gh is not in D , the vertex gh must be dominated by a vertex in D . Therefore, either $gh' \in D$ with $(h, h') \in E(H)$, or $g'h \in D$ with $(g, g') \in E(G)$. In first case, the edge $(gh', gh) \in E(C_{r+1}^{nn'}) \cap E(G \square H)$, and in

the second case, the edge $(g'h, gh) \in E(C_{r+1}^{nn'}) \cap E(G \square H)$. In either case, $E(C_{r+1}^{nn'}) \cap E(G \square H) \neq \emptyset$, and each of the cover equations associated with $G \square H$ are satisfied.

Conversely, assume that each of the cover equations associated with $G \square H$ are satisfied. We must show that $G \square H$ has a dominating set of size r . We proceed by contradiction. Assume that $G \square H$ does not have a dominating set of size r , and consider the complement of $G \square H$ (denoted by $\overline{G \square H}$).

Since $G \square H$ does not contain a dominating set of size r , every subset of size r in $V(\overline{G \square H})$ has a common neighbor. Thus, by Definition 4.5, $\overline{G \square H}$ is a $r + 1$ -cover. Furthermore, $E(G \square H) \cap E(\overline{G \square H}) = \emptyset$. We remove edges from $\overline{G \square H}$ until we find a subgraph C that is a minimal $r + 1$ -cover. Since C is a subgraph of $\overline{G \square H}$, $E(G \square H) \cap E(C) = \emptyset$. However, edges in C of the form $(gh, g'h')$ where $g \neq g'$ and $h \neq h'$ do not correspond to variables in our system of polynomial equations, and thus, we have not yet shown that the set $\cap_{\square} C$ is *non-empty* and the equation $P(\cap_{\square} C) = 0$ is *not* satisfied. We must show that C contains at least one edge of the form $(gh, g'h)$ or (gh, gh') .

By assumption, $r \geq \min(n, n')$. Without loss of generality, let $n' = |V(H)| \geq n = |V(G)|$ and let $S \subseteq V(C) = V(\overline{G \square H})$ be a subset of size r such that $\{g : gh \in S\} = V(G)$. In other words, choose S such that there is at least one vertex in S per G -level. Since C is an $r + 1$ -cover, S has a common neighbor $gh \in V(C)$, and since gh is connected to every vertex in S and there is at least one vertex per G -level in S , there exists a vertex $gh' \in S$. Thus, $(gh, gh') \in E(C)$, which implies that the set $\cap_{\square} C$ is *non-empty*. But recall that $E(G \square H) \cap E(C) = \emptyset$, which implies that the equation $P(C) = 0$ is *not* satisfied. This is a contradiction with our assumption that each of the cover equations associated with minimal $r + 1$ -covers are satisfied. Thus, $G \square H$ must contain a dominating set of size r . \square

The ideal described by Theorem 4.19 is the set of polynomials vanishing on graphs G, H with dominating sets k, l respectively, such that $G \square H$ has a dominating set of size r . Thus, the ideal described by Theorem 4.19 is the same as the ideal described by Theorem 4.3: both are $I(n, k, n', l, r)$. However, the question of whether the basis described by Theorem 4.3 is a universal Gröbner basis is not immediately resolvable. Given two $r + 1$ -covers C and C' on nn' vertices, it does seem possible that for large values of r and smaller values of k and l , a $k + 1$ -cover on n vertices (or an $l + 1$ -cover on n' vertices) might be a proper subgraph of $C \cap C'$. In such a scenario, the collection of linear factors would *not* satisfy the linear factor criterion described by Definition 3.2, and the basis would *not* be a universal Gröbner basis. The natural question to then pose is the following: for what values of r is the cover representation of $I(n, k, n', l, r)$ a universal Gröbner basis? In Section 5, we will show that this question is equivalent to Vizing's conjecture.

5 An Algebraic Representation of Vizing's Conjecture

We now express Vizing's conjecture in terms of the ideals described in Section 4. By defining two particular zero-dimensional, radical ideals and then intersecting them, we transform Vizing's conjecture from a question about dominating sets and product graphs to an ideal membership question involving the product of two polynomials. This algebraic representation suggests a variety of computational approaches, from large-scale, sparse linear algebra computations such as those demonstrated in [21, 20, 23] to Gröbner basis calculations customized for the linear factor criterion (Def. 3.2). Although Gröbner basis computations and the ideal membership question itself are both known to be EXPSPACE-complete [13], algebraic algorithms customized for specific NP-complete combinatorial properties (such as graph 3-colorability) have yielded surprisingly practical compu-

tation approaches on *some* large examples [20]. However, refinements to known algorithms based on the algebraic and combinatorial properties of these particular dominating set representations have not yet been explored. We leave this question for future work.

After describing the algebraic representation of Vizing's conjecture, we then transfer the ideal membership question involving the product of two cover polynomials back to the realm of graph theory. This transformation yields a conjecture involving k -covers that turns out to be the complement of Vizing's conjecture. We conclude by reclaiming a known result with a cover-based proof, and also by linking universal Gröbner basis to Vizing's conjecture.

5.1 Ideals, Varieties and Vizing's Conjecture

We begin by defining three different varieties, and then relating the vanishing ideals associated with these varieties to the ideals described in Section 4. Each of the following varieties are a set of points such that each point represents a pair of graphs G, H in n, n' vertices such that $G \square H$ has a dominating set of size $kl - 1$. Additionally,

1. Let V_l^{k-1} be a variety such that every point represents a pair of labeled graphs G, H with dominating sets of size $k - 1, l$ respectively.
2. Let V_{l-1}^k be a variety such that every point represents a pair of labeled graphs G, H with dominating sets of size $k, l - 1$ respectively.
3. Let V_l^k be a variety such that every point represents a pair of labeled graphs G, H with dominating sets of size k, l respectively.

Throughout this section, graphs are always labeled graphs. We now define the following ideals.

$$I_l^{k-1} := I(n, k - l, n', l, kl - 1) , \quad I_{l-1}^k := I(n, k, n', l - l, kl - 1) , \quad I_l^k := I(n, k, n', l, kl - 1) .$$

Notice that in each of these ideals, we have set $r = kl - 1$. According to the definitions of $I(n, k, n', l, r)$ given in Section 4, $V_l^k = V(I_l^k)$, $V_l^{k-1} = V(I_l^{k-1})$ and $V_{l-1}^k = V(I_{l-1}^k)$. Recall that if a given ideal I is radical, then $I(V(I)) = I$. Since the ideals I_l^k, I_l^{k-1} and I_{l-1}^k are radical (by Lemma 2.1), regardless of whether we choose the representation of I_l^k from Section 4.1, or the representation of I_l^k from Section 4.3, $I_l^k = I(V_l^k)$, $I_l^{k-1} = I(V_l^{k-1})$ and $I_{l-1}^k = I(V_{l-1}^k)$. In the following lemmas, we will relate Vizing's conjecture to these ideals and varieties. It is important to note that the algebraic representation of Vizing's conjecture described below is *independent* of the internal representation of the ideal $I(n, k, n', l, r)$. In other words, the following lemmas will hold for *any* representation of I_l^k, I_l^{k-1} and I_{l-1}^k as long as the ideals are radical, and $I_l^k = I(V_l^k)$, $I_l^{k-1} = I(V_l^{k-1})$ and $I_{l-1}^k = I(V_{l-1}^k)$. Thus, if another representation of $I(n, k, n', l, r)$ is found in the future with a differing set of computational or combinatorial properties, this formulation of Vizing's conjecture remains valid.

Lemma 5.1 $V_l^{k-1} \cup V_{l-1}^k = V_l^k$ if and only if then Vizing's conjecture is true.

Proof: Every point in V_l^k corresponds to a graph G with $\gamma(G) \leq k$ and a graph H with $\gamma(H) \leq l$ such that $\gamma(G \square H) \leq kl - 1$.

Assume that Vizing's conjecture is true. Thus, $\gamma(G)\gamma(H) \leq \gamma(G \square H)$ for all graphs G, H . In particular, if $\gamma(G) = k$ and $\gamma(H) = l$, then $kl \leq \gamma(G \square H)$. In particular, $\gamma(G \square H)$ can never equal

$kl-1$. Thus, if there is a pair of graphs G, H such that $\gamma(G \square H) \leq kl-1$, then either $\gamma(G) \leq k-1$ or $\gamma(H) \leq l-1$. Thus, any point appearing in V_l^k also appears in $V_l^{k-1} \cup V_{l-1}^k$, and $V_l^{k-1} \cup V_{l-1}^k = V_l^k$.

Conversely, assume that $V_l^{k-1} \cup V_{l-1}^k = V_l^k$. Then, for every pair of graphs G, H such that $\gamma(G \square H) \leq kl-1$, then either $\gamma(G) \leq k-1$ or $\gamma(H) \leq l-1$. In other words, if $\gamma(G) = k$ and $\gamma(H) = l$, then $\gamma(G \square H) > kl-1$. This implies that $\gamma(G)\gamma(H) \leq \gamma(G \square H)$, or that Vizing's conjecture is true. \square

Lemma 5.2 $I_l^{k-1} \cap I_{l-1}^k = I_l^k$ if and only if Vizing's conjecture is true.

Proof: If $I_l^{k-1} \cap I_{l-1}^k = I_l^k$, then $V_l^{k-1} \cup V_{l-1}^k = V_l^k$ (and vice versa). This can be seen via the results in Section 2, since if $I_l^{k-1} \cap I_{l-1}^k = I_l^k$, then $V(I_l^{k-1} \cap I_{l-1}^k) = V(I_l^{k-1}) \cup V(I_{l-1}^k) = V_l^{k-1} \cup V_{l-1}^k = V_l^k$. Thus, Vizing's conjecture is true via Lemma 5.1. \square

We have now equated Vizing's conjecture with a question about the equality of two ideals. To prove this equality, we must prove the inclusion in both directions. The inclusion in one direction is fairly easy to see; however, the inclusion in the other direction remains open.

Lemma 5.3 $V_l^{k-1} \cup V_{l-1}^k \subseteq V_l^k$.

Proof: Consider any point p in $V_l^{k-1} \cup V_{l-1}^k$. This point either represents graphs G, H with dominating sets $k-1, l$ respectively, or graphs G, H with dominating sets $k, l-1$ respectively (in both cases, $G \square H$ contains a dominating set of size $kl-1$). However, if G contains a dominating set of size $k-1$ (or $l-1$), G also contains a dominating set of size k (or size l). Therefore, any point representing graphs G, H with dominating sets $k-1, l$ respectively, or graphs G, H with dominating sets $k, l-1$ respectively also represents graphs G, H with dominating sets k, l respectively. Thus, any point $p \in V_l^{k-1} \cup V_{l-1}^k$ is also a point $p \in V_l^k$. \square

Corollary 5.4 $I_l^k \subseteq I_l^{k-1} \cap I_{l-1}^k$.

Proof: Since $V_l^{k-1} \cup V_{l-1}^k \subseteq V_l^k$, this implies that $I(V_l^k) \subseteq I(V_l^{k-1} \cup V_{l-1}^k)$, which implies that $I_l^k \subseteq I_l^{k-1} \cap I_{l-1}^k$ (see [8], Chapter 4). \square

Thus, in order to conclude a proof Vizing's conjecture, we must show that $I_l^{k-1} \cap I_{l-1}^k \subseteq I_l^k$. However, recall from the conclusion of Section 2, if the representations of I_l^k, I_l^{k-1} and I_{l-1}^k satisfy the conditions specified in Lemma 2.2, we can explicitly describe a basis for $I_l^{k-1} \cap I_{l-1}^k$:

$$I_l^{k-1} \cap I_{l-1}^k = \sqrt{I_l^{k-1} \cdot I_{l-1}^k} = \langle fg : f \in I_l^{k-1}, g \in I_{l-1}^k \rangle + \langle e^2 - e \rangle.$$

Thus far in this section, without specifying the representation for I_l^k , etc., we have shown that proving Vizing's conjecture is equivalent to proving the equality of two ideals. We will now show that for each of the representations specified in Section 4, proving the opposite inclusion reduces to a problem in ideal membership, which implies, far more significantly, that proving Vizing's conjecture reduces to a computational problem in Gröbner bases or linear algebra.

Using the Representation from Theorem 4.3:

Let h be any polynomial in $I_l^{k-1} \cap I_{l-1}^k$, where I_l^{k-1} and I_{l-1}^k are represented using the system of polynomials defined by Theorem 4.1. Then $h \in I_l^{k-1} \cap I_{l-1}^k$ can be written as follows:

$$\begin{aligned} h = & (\cdot)P_G^{k-1}P_G^k + (\cdot)P_G^{k-1}P_H^{l-1} + (\cdot)P_G^{k-1}P_{G \square H} + (\cdot)P_G^{k-1}(e^2 - e) + (\cdot)P_H^lP_G^k + (\cdot)P_H^lP_H^{l-1} \\ & + (\cdot)P_H^lP_{G \square H} + (\cdot)P_H^l(e^2 - e) + (\cdot)P_{G \square H}P_G^k + (\cdot)P_{G \square H}P_H^{l-1} + (\cdot)P_{G \square H}P_{G \square H} + (\cdot)P_{G \square H}(e^2 - e) \\ & + (\cdot)(e^2 - e)P_G^k + (\cdot)(e^2 - e)P_H^{l-1} + (\cdot)(e^2 - e)P_{G \square H} + (\cdot)(e^2 - e)(e^2 - e) + (\cdot)(e^2 - e), \end{aligned} \quad (8)$$

where $P_{G \square H}$ is equal to $P_{G \square H}^{kl-1}$. We must show that h is also in I_l^k . Specifically, we must show that there exists polynomial coefficients such that

$$h = (\cdot)P_G^k + (\cdot)P_H^l + (\cdot)P_{G \square H} + (\cdot)(e^2 - e). \quad (9)$$

Comparing Eqs. 8 and 9, we see that the only term which is not *already* expressed in terms of polynomials in I_l^k is the product $P_G^{k-1}P_H^{l-1}$. Thus, we must find an algebraic relationship or a *syzygy* such that

$$P_G^{k-1}P_H^{l-1} = (\cdot)P_G^k + (\cdot)P_H^l + (\cdot)P_{G \square H} + (\cdot)(e^2 - e). \quad (10)$$

This is equivalent to asking whether or not $P_G^{k-1}P_H^{l-1} \in I_l^k$. We note this link between Vizing's conjecture and ideal membership: a conjecture about dominating sets in product graphs reduces to an ideal membership question concerning the product of two polynomials.

Lemma 5.5 $P_G^{k-1}P_H^{l-1} \in I_l^k$ if and only if Vizing's conjecture is true.

Proof: If $P_G^{k-1}P_H^{l-1} \in I_l^k$, then any polynomial $h \in I_l^{k-1} \cap I_{l-1}^k$ is also in I_l^k . In other words, $I_l^{k-1} \cap I_{l-1}^k \subseteq I_l^k$. Lemma 5.3 and Corollary 5.4 establish the other direction of the inclusion, and $I_l^{k-1} \cap I_{l-1}^k = I_l^k$. Thus, by Lemma 5.2, Vizing's conjecture is true. Conversely, if Vizing's conjecture is true, then $I_l^{k-1} \cap I_{l-1}^k = I_l^k$, which implies that $P_G^{k-1}P_H^{l-1} \in I_l^k$. \square

Unfortunately, performing these computations was quite difficult. We worked on machines with dual Opteron nodes, 2 GHz clock speed, and 12 GB of RAM. We used the custom C++ exact arithmetic linear algebra solver and the method described in [21], [20] and [23]. We first tested the smallest possible example of $n = k = n' = l = 2$. In order to find the syzygy defined by Eq. 10, we constructed a 100×84 linear system, solved in under a second, yielding a syzygy of degree 6. However, when we tested the next largest example, with $n = 3, k = n' = l = 2$, we were unable to find a syzygy, although we determined that the syzygy had degree 8 or larger. The degree 7 linear algebra system had size $158,412 \times 2,310$ and took 6,862 seconds (≈ 2 hours) to solve. However, the degree 8 linear algebra system remained unsolved even after days of computation, most likely due to the magnitude of the numbers involved. For example, one of the coefficients in P_G^k was 2,106,048,060. It is easy to imagine the computationally intensive nature of exact arithmetic on numbers of that order in a linear system with millions of rows and columns. However, we again note that the specific combinatorial properties of these representations have not yet been exploited. It is a question of future work to improve these computations.

Using the Representation from Theorem 4.19:

In Lemma 5.5, we showed that proving Vizing's conjecture using the representation described by Theorem 4.3 was equivalent to proving that $P_G^{k-1}P_H^{l-1} \in I_l^k$. We now show that when the

representation of I_l^k corresponding to covers (Theorem 4.19) is used, Vizing's conjecture reduces to both an ideal membership question involving the product of two cover polynomials, and a graph theoretic question that is the complement of Vizing's conjecture.

In the cover representation of I_l^k , we denote the various cover polynomials as $P(C_k^n), P(C_l^{n'})$ and $P(\cap_{\square} C_{kl}^{nn'})$. As before, $h \in I_l^{k-1} \cap I_{l-1}^k$ can be written as follows:

$$\begin{aligned}
h = & (\cdot)P(C_k^n)P(C_{k+1}^n) + (\cdot)P(C_k^n)P(C_l^{n'}) + (\cdot)P(C_k^n)P(\cap_{\square} C_{kl}^{nn'}) + (\cdot)P(C_k^n)(e^2 - e) \\
& + (\cdot)P(C_{l+1}^{n'})P(C_{k+1}^n) + (\cdot)P(C_{l+1}^{n'})P(C_l^{n'}) + (\cdot)P(C_{l+1}^{n'})P(\cap_{\square} C_{kl}^{nn'}) + (\cdot)P(C_{l+1}^{n'})(e^2 - e) \\
& + (\cdot)P(\cap_{\square} C_{kl}^{nn'})P(C_{k+1}^n) + (\cdot)P(\cap_{\square} C_{kl}^{nn'})P(C_l^{n'}) + (\cdot)P(\cap_{\square} C_{kl}^{nn'})P(\cap_{\square} C_{kl}^{nn'}) \\
& + (\cdot)P(\cap_{\square} C_{kl}^{nn'})(e^2 - e) + (\cdot)(e^2 - e)P(C_{k+1}^n) + (\cdot)(e^2 - e)P(C_l^{n'}) + (\cdot)(e^2 - e)P(\cap_{\square} C_{kl}^{nn'}) \\
& + (\cdot)(e^2 - e)(e^2 - e) + (\cdot)(e^2 - e). \tag{11}
\end{aligned}$$

Again, to conclude the proof of Vizing's conjecture, we must show that $h \in I_l^k$. Specifically, we must show that there exist coefficients such that

$$h = (\cdot)P(C_{k+1}^n) + (\cdot)P(C_{l+1}^{n'}) + (\cdot)P(\cap_{\square} C_{kl}^{nn'}) + (\cdot)(e^2 - e). \tag{12}$$

Comparing Eqs. 11 and 12, we see that the only term which is not *already* expressed in terms of polynomials in I_l^k is the product $P(C_k^n)P(C_l^{n'})$. Thus, we must find an algebraic relationship or a *syzygy* such that

$$P(C_k^n)P(C_l^{n'}) = (\cdot)P(C_{k+1}^n) + (\cdot)P(C_{l+1}^{n'}) + (\cdot)P(\cap_{\square} C_{kl}^{nn'}) + (\cdot)(e^2 - e).$$

This is equivalent to asking whether or not $P(C_k^n)P(C_l^{n'}) \in I_l^k$. However, recall that the cover representations are not just a basis, but a universal Gröbner basis. Our experimental investigations using CoCoA Lib [1] have indicated that not only is $P(C_k^n)P(C_l^{n'}) \in I_l^k$, but that there exists a $P(\cap_{\square} C_{kl}^{nn'})$ such that $P(C_k^n)P(C_l^{n'}) = P(\cap_{\square} C_{kl}^{nn'})$. In other words, $P(C_k^n)P(C_l^{n'})$ is itself an element of the Gröbner basis of I_l^k . We were not able to verify this conjecture on large examples because of the exponential number of monomials in the expanded cover polynomials. For example, $P(C_k^n)$ may be compactly represented as a product of L linear factors, but when $P(C_k^n)$ is expanded during the calculation of the Gröbner basis, 2^L monomials are generated. Indeed, even very small examples often took days of computation. We are interested in exploring modifications to Gröbner basis algorithms to exploit the factored form of these input bases for future work.

We will now prove several lemmas and extrapolate the conjecture that there exists a polynomial $P(\cap_{\square} C_{kl}^{nn'})$ such that $P(C_k^n)P(C_l^{n'}) = P(\cap_{\square} C_{kl}^{nn'})$ to a very specific graph theory conjecture. Recall that $\cap_{\square} C_{kl}^{nn'}$ (Definition 4.18) denotes a specific set of edges.

Lemma 5.6 *Given a minimal k -cover C_k^n and a minimal l -cover $C_l^{n'}$, if there exists a (kl) -cover $C_{kl}^{nn'}$ such that $E(C_k^n \square C_l^{n'}) = \cap_{\square} C_{kl}^{nn'}$, then Vizing's conjecture is true.*

Proof: If there exists a (kl) -cover $C_{kl}^{nn'}$ such that $E(C_k^n \square C_l^{n'}) = \cap_{\square} C_{kl}^{nn'}$, then $P(C_k^n)P(C_l^{n'}) = P(\cap_{\square} C_{kl}^{nn'})$, and $P(C_k^n)P(C_l^{n'}) \in I_l^k$. Thus, $I_l^{k-1} \cap I_{l-1}^k = I_l^k$, and Vizing's conjecture is true. \square

We will now define a product graph that specifically relates to the \cap_{\square} intersection.

Definition 5.7 *Given graphs G and H , the star product $G \star H$ has vertex set $V(G) \times V(H)$ and edge set $E(G \star H) = E(G \square H) \cup \{(gh, g'h') : g \neq g' \text{ and } h \neq h'\}$.*

Proposition 5.8 *Given a k -cover C_k^n and a l -cover $C_l^{n'}$, then $E(C_k^n \square C_l^{n'}) = \cap_{\square} (C_k^n \star C_l^{n'})$.*

Proof: This follows directly from Definitions 4.18 and 5.7. \square

We note that $C = C_k^n \star C_l^{n'}$ contains the largest amount of edges such that $\cap_{\square} C = C_k^n \square C_l^{n'}$. The question that remains is the following.

Conjecture 5.9 *Given minimal k, l -covers $C_k^n, C_l^{n'}$, $C_k^n \star C_l^{n'}$ is a kl -cover.*

This is the complement of Vizing's conjecture. We observe that we can easily prove two of the known properties of covers on $C_k^n \star C_l^{n'}$.

Proposition 5.10 *Given minimal k, l -covers $C_k^n, C_l^{n'}$, every vertex in $V(C_k^n \star C_l^{n'})$ is contained in a kl -clique.*

Proof: Let gh be a vertex in $C_k^n \star C_l^{n'}$. We must show that gh appears in a kl -clique. Since $C_k^n, C_l^{n'}$ are a k, l -covers respectively, g appears in a k -clique in C_k^n , and h appears in an l -clique in $C_l^{n'}$. Let $g \cup \{g_1, \dots, g_{k-1}\}$ be the k -clique in C_k^n , and let $h \cup \{h_1, \dots, h_{l-1}\}$ be the l -clique in $C_l^{n'}$. We claim

$$Q = (g \cup \{g_1, \dots, g_{k-1}\}) \times (h \cup \{h_1, \dots, h_{l-1}\})$$

is a kl -clique in $C_k^n \star C_l^{n'}$. Let gh and $g'h'$ be two vertices in Q . If $g = g'$, then $(h, h') \in C_l^{n'}$, since h, h' appear in an l -clique. If $h = h'$, then $(g, g') \in C_k^n$, since g, g' appear in an k -clique. In both cases, $(gh, g'h')$ is an edge in $C_k^n \star C_l^{n'}$. Finally, if $g \neq g'$ and $h \neq h'$, then $(gh, g'h')$ is an edge in $C_k^n \star C_l^{n'}$ by the definition of the star product. Thus, we have shown that Q is a kl -clique. \square

Additionally, we can see that if k and l are strictly greater than two, then $C_k^n \star C_l^{n'}$ has diameter at most two. This can be seen by choosing any two vertices $gh, g'h'$ in $C_k^n \star C_l^{n'}$, and noting that there must exist a third vertex $g''h''$ such that $g''h''$ is adjacent to both gh and $g'h'$. Thus, under these conditions, the diameter of $C_k^n \star C_l^{n'}$ is at most two.

When framing Vizing's conjecture in terms of covers and the star product, we can easily reclaim the complement of a known result by El-Zahar and Pareek [9].

Theorem 5.11 *Given minimal k, l -covers $C_k^n, C_l^{n'}$, such that $kl - 1 < \min(n, n')$, then $C_k^n \star C_l^{n'}$ is a kl -cover.*

Proof: Consider any set S of $kl - 1$ vertices in $C_k^n \star C_l^{n'}$. Let P_k^n denote C_k^n -projection of S (the set of g coordinates such that gh is a vertex in S), and let $P_l^{n'}$ denote $C_l^{n'}$ -projection of S (the set of h coordinates such that gh is a vertex in S). Since $kl - 1 < \min(n, n')$, $|P_k^n|, |P_l^{n'}| < \min(n, n')$. Thus, there exists a vertex $g'h'$ such that $g \neq g'$ and $h \neq h'$. Thus, gh is adjacent to $g'h'$, and $g'h'$ is the common neighbor of S . Since S was a general set, $C_k^n \star C_l^{n'}$ is a kl -cover. \square

Since we have shown that $C_k^n \star C_l^{n'}$ is a kl -cover, by Lemma 5.6, we have shown that Vizing's conjecture holds on this class of graphs. This is the complement of the result proven in [9]. The restriction placed on the value of r in Theorem 4.19 now becomes clear. We only define the cover representation of I_l^k when $kl - l \geq \min(n, n')$. But since Vizing's conjecture holds whenever $kl - 1 < \min(n, n')$, the method of intersecting ideals is defined only on the as-yet unproven cases of Vizing's conjecture.

We conclude by drawing a parallel between universal Gröbner bases and Vizing's conjecture based on our experimental investigations with CoCoA Lib [1]. Since a $k + 1$ -cover is not a proper

subset of a k -cover, and an $l + 1$ -cover is not a proper subset of a l -cover, if any minimal kl -cover can be written as $C_k^n \star C_l^{n'}$ for some minimal $C_k^n, C_l^{n'}$, then not only is Vizing's conjecture true, but replacing

$$P(\cap_{\square} C) = 0, \quad \text{for each } C \in \mathcal{C}_{kl}^{nn'},$$

with

$$P(C_k^n \star C_l^{n'}) = 0, \quad \text{for each } C_k^n \in \mathcal{C}_k^n \text{ and each } C_l^{n'} \in \mathcal{C}_l^{n'},$$

in Theorem 4.19 would yield a universal Gröbner basis for $I(n, k, n', l, r = kl - 1)$.

Acknowledgements

The authors would like to thank Jon Lee at the IBM T.J. Watson Research Center for arranging our use of the Yellowzone IBM machines.

References

- [1] J. Abbott and A.M. Bigatti. CoCoALib: a C++ library for doing Computations in Commu-
tative Algebra. Available at <http://cocoa.dima.unige.it/cocoalib>.
- [2] N. Alon. Combinatorial Nullstellensatz. *Combinatorics, Prob. and Computing*, 8:7–29, 1999.
- [3] N. Alon and M. Tarsi. Colorings and orientations of graphs. *Combinatorica*, 12:125–134, 1992.
- [4] E. Babson, S. Onn, and R. Thomas. The Hilbert zonotope and a polynomial time algorithm
for universal Gröbner bases. *Advances in Applied Mathematics*, 30(3):529–544, 2003.
- [5] A. Barcalkin and L. German. The external stability number of the cartesian product of graphs.
Bul. Akad. Stiinte RSS Moldoven., 1(94):5–8, 1979.
- [6] B.Stjan, B.Brešar, P.Dorbec, W.Goddard, B.Hartnell, M.Henning, S.Klavžar, and D.Rall.
Vizing's conjecture: A survey and recent results, 2009. preprint.
- [7] W. Clark and S. Suen. An inequality related to Vizing's conjecture. *Electronic Journal of
Combinatorics*, 7(Note 4), 2000.
- [8] D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties and Algorithms*. Springer, New York, 1998.
- [9] M. El-Zahar and C.M. Pareek. Domination number of products of graphs. *Ars Combin.*,
31:223–227, 1991.
- [10] S. Eliahou. An algebraic criterion for a graph to be four-colourable. *Aportaciones Matemáticas*,
6:3–27, 1992.
- [11] K.G. Fischer. Symmetric polynomials and Hall's theorem. *Discrete Math*, 69(3):225–234, 1988.
- [12] M. Garey and D. Johnson. *Computers and Intractability: A Guide to the Theory of NP-
Completeness*. W.H. Freeman and Company, 1979.

- [13] J. Gathen and J. Gerhard. *Modern Computer Algebra*. Cambridge University Press, Cambridge, second edition, 1999.
- [14] B. Hartnell and D. Rall. Domination in cartesian products: Vizing’s conjecture. In *Domination in graphs*, pages 163–189, New York, 1998. Mono. Textbooks Pure and Appl. Math, Dekker.
- [15] C.J. Hillar and T. Windfeldt. An algebraic characterization of uniquely vertex colorable graphs. *Journal of Combinatorial Theory Series B*, 98:400–414, 2008.
- [16] M. Kreuzer and L. Robbiano. *Computational Commutative Algebra I*. Springer-Verlag, 2000.
- [17] S.R. Li and W.W. Li. Independence number of graphs and generators of ideals. *Combinatorica*, 1:55–61, 1981.
- [18] J.A. De Loera. Gröbner bases and graph colorings. *Beitrage zur Algebra und Geometrie*, 36(1):89–96, 1995.
- [19] J.A. De Loera, C. Hillar, P.N. Malkin, and M. Omar. Recognizing graph theoretic properties with polynomial ideals. *Elect. J. of Combinatorics.*, 17(1):R114, 2010.
- [20] J.A. De Loera, J. Lee, P.N. Malkin, and S. Margulies. Hilbert’s Nullstellensatz and an algorithm for proving combinatorial infeasibility. In *Proceedings of the twenty-first international symposium on Symbolic and algebraic computation*, pages 197–206. ISSAC, 2008.
- [21] J.A. De Loera, J. Lee, S. Margulies, and S. Onn. Expressing combinatorial optimization problems by systems of polynomial equations and the Nullstellensatz. *Combinatorics, Probability and Computing*, 18(4):551–582, 2009.
- [22] L. Lovász. Stable sets and polynomials. *Discrete Mathematics*, 124:137–153, 1994.
- [23] S. Margulies. Computer Algebra, Combinatorics, and Complexity: Hilbert’s Nullstellensatz and NP-Complete Problems. *University of California at Davis, Ph.D. Thesis*, 2008.
- [24] Y. Matiyasevich. A criteria for colorability of vertices stated in terms of edge orientations (in russian). *Discrete Analysis (Novosibirsk)*, 26:65–71, 1974.
- [25] Y. Matiyasevich. Some algebraic methods for calculation of the number of colorings of a graph (in russian). *Zapiski Nauchnykh Seminarov POMI*, 293:193–205, 2001.
- [26] K. Mizuno and S. Nishihara. Nowhere-zero flow polynomials. *Journal of Combinatorial Theory, Series A*, 108(2):205–215, 2004.
- [27] M. Mnuk. Representing graph properties by polynomial ideals. In *Computer Algebra in Scientific Computing*, pages 431–444. 4th Inter. Work. on Comp. Alg. in Sci. Comp., 2001.
- [28] A. Simis, W. Vasconcelos, and R. Villarreal. On the ideal theory of graphs. *Journal of Algebra*, 167(2):389–416, 1994.
- [29] D. Sumner and P. Blitch. Domination critical graphs. *Journal of Combinatorial Theory, Series B*, 34:65–76, 1983.
- [30] L. Sun. A result on Vizing’s conjecture. *Discrete Mathematics*, 275(1):363–366, 2004.
- [31] V. Vizing. Some unsolved problems in graph theory. *Uspekhi Mat. Nauk*, 23:117–134, 1968.