Graph-coloring ideals: Nullstellensatz certificates, Gröbner bases for chordal graphs, and hardness of Gröbner bases

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ABSTRACT

We consider a well-known family of polynomial ideals encoding the problem of graph-k-colorability. Our paper describes how the inherent combinatorial structure of the ideals implies several interesting algebraic properties. Specifically, we provide lower bounds on the difficulty of computing Gröbner bases and Nullstellensatz certificates for the coloring ideals of general graphs. We revisit the fact that computing a Gröbner basis is NP-hard and prove a robust notion of hardness derived from the inapproximability of coloring problems. For chordal graphs, however, we explicitly describe a Gröbner basis for the coloring ideal and provide a polynomial-time algorithm to construct it.

Categories and Subject Descriptors

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1. INTRODUCTION

Many authors in computer algebra and complexity theory have studied the complexity of Gröbner bases (see e.g., [19, 23, 34, 35, 39] and references therein) and the difficulty of Hilbert's Nullstellensatz (see [5, 6, 7, 12, 28, 29, 33]). With few exceptions authors have concentrated on proving worst-case upper bounds. In this paper we look instead at the behavior of Gröbner bases and Hilbert Nullstellensätze in a concrete combinatorial family of polynomials. Our key point is to study how the structure of graph coloring problems provides lower bounds on the difficulty of finding Gröbner bases and Nullstellensatz certificates, providing a counterpart to lower and upper bound theorems for general polynomial systems.

Many authors have studied the rich connection between graphs and polynomials (see e.g., [2, 3, 13, 26, 30, 32, 37]). Our starting point is Bayer's theorem for 3-colorings [4], further generalized in [15, 17] to k-coloring over a finite field. Suppose we wish to check whether a graph $G = (V, E)$ is k-colorable, and set $n = |V|$. For fields $\mathbb{K}$ of characteristic not dividing k, we define the *k-coloring ideal* $\mathcal{I}_k(G) \subset \mathbb{K}[x_1, \ldots, x_n]$ (also denoted by $\mathcal{I}_G$ if the number of colors is clear) to be the ideal generated by the *vertex polynomials* $\nu_i := x_i^k - 1$, for $1 \leq i \leq n$, and the *edge polynomials* $\eta_{i,j} := \sum_{l=0}^{k-1} x_i^l x_j^{k-1-l}$, for $\{i, j\} \in E$. The set of all vertex and edge polynomials of a graph G is denoted by $\mathcal{F}_G$.

THEOREM 1.1 (SEE [15, 17]). *The graph G is k-colorable if and only if $\mathcal{I}_k(G)$ has a common root in the algebraic closure of $\mathbb{K}$. In other words, G is not k-colorable if and only if $\mathcal{I}_k(G) = \langle 1 \rangle = \mathbb{K}[x_1, \ldots, x_n]$. Moreover, the dimension of the vector space $\mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_k(G)$ equals $k!$ times the number of distinct k-colorings of G.*

The ideal $\mathcal{I}_k(G)$ has been used for interesting applications to graph theory (see [26] and references there). From the well-known Hilbert Nullstellensatz [10], one can derive *certificates* that a system of polynomials has no solution (i.e., in our case, that a graph does not have a k-coloring).

THEOREM 1.2 (HILBERT'S NULLSTELLENSATZ [10]). *Suppose that* $f_1, \ldots, f_m \in \mathbb{K}[x_1, \ldots, x_n]$. *Then, there are no solutions to the system* $\{f_i = 0\}$ *in the algebraic closure of* $\mathbb{K}$, *if and only if there exist* $\alpha_i \in \mathbb{K}[x_1, \ldots, x_n]$ *such that*

$$\alpha_1 f_1 + \cdots + \alpha_m f_m = 1.$$

We refer to the set $\{\alpha_i\}$ as a *Nullstellensatz certificate*, and measure the complexity of a certificate by its *degree*, defined as the maximum degree of any α_i. If a system is known to have a Nullstellensatz certificate of small constant degree (over a finite field), one can simply find this certificate by a series of linear algebra computations [14, 15, 16]. There are well-known *upper bounds* for the degrees of the coefficients α_i in the Nullstellensatz certificate for *general* systems of polynomials that grow with the number of variables [28]. Furthermore, these bounds turn out to be sharp for some pathological instances.

Connections between complexity theory and the Gröbner bases and Nullstellensätze of coloring ideals have been made in [5, 30, 31]. These include the result that unless NP = coNP, there must exist an infinite family of non-3-colorable graphs for which the minimum degree of a Hilbert Nullstellensatz certificate grows arbitrarily large [16, 31]. We remark that for an ideal I for which the set of zeroes is finite, the vector space dimension of the quotient ring obtained from I is finite, too (see §2.2 of [11]). Zero-dimensional ideals admit special algorithms (see e.g. [20]). Cifuentes and Parrilo [9] identify graph structure within an arbitrary polynomial system and show that this yields faster algorithms for solving systems of polynomials. Our coloring ideals are quite unique as they both have unique structure inherent from the graph relating the polynomials and are zero-dimensional.

This article offers three new contributions in the structure and complexity of Gröbner bases and Nullstellensätze of coloring ideals.

(1) In Section 2.1, we show that the minimal degree of Nullstellensatz certificates of coloring ideals satisfies certain modular constraints and that it grows at least linearly in the number of colors (similar results were observed for other ideals in [8]). We indicate that the field of coefficients has some intriguing influence upon the complexity and propose a conjecture.

(2) It is well-known that many combinatorial problems are hard to solve even approximately. For instance, Khanna et al. [27] have shown it is NP-hard to 4-color a 3-colorable graph. More strongly, even if one is allowed to ignore a particular small (but non-zero) fraction of nodes, it is NP-hard to properly 4-color the remaining nodes.

In Section 2.2, we demonstrate how one can transfer inapproximability results for graphs to inapproximability results for polynomial rings. We prove that it is hard to compute a Gröbner basis for an ideal even if we are allowed to ignore a large subset of the generators for our ideal. Thus the coloring ideal provides a sense of "robust" hardness for the computation of Gröbner bases.

(3) Despite hardness in the general case of computing a Gröbner basis, we might hope that some algorithm could find Gröbner bases efficiently, particularly if we restrict our focus to some special class of systems of polynomials. In Section 3, we prove that computing a Gröbner basis can be done in polynomial time when the associated graph is *chordal*. We describe explicitly the structure of such a Gröbner basis.

For background on the material presented in this paper, we direct the interested reader to the books [1, 10, 11, 22].

2. LOWER BOUNDS ON HARDNESS: GRÖBNER BASES & NULLSTELLENSÄTZE

Deciding whether a graph is k-colorable is an NP-complete problem, and we encoded it as the solution of a multivariate polynomial system (see e.g., [4, 17, 13]). It is clear that if the system of equations in the coloring ideal can be solved in polynomial time (in the input size) for 3-coloring ideals, then P = NP. What makes this very interesting is that one can see (or at least try to see) algebraic phenomena that are produced by the separation of complexity classes. For example, assuming that P $\neq$ NP then the degree of Nullstellensatz certificates for systems of equations coming from non-3-colorable graphs must show some growth in the degree. Now we discuss two ways in which the hardness of solving the coloring problem algebraically is made concrete. We note that in [8], prior similar work was done on a different family of zero-dimensional ideals.

2.1 Nullstellensätze

In this section, we consider $N_{k,\mathbb{K}}(G)$, the minimal Nullstellensatz degree for the k-coloring ideal of a graph G over the field $\mathbb{K}$. We show that $N_{k,\mathbb{K}}(G)$ grows at least linearly with respect to k, and provide evidence that the growth is, in fact, faster. Note that $N_{k,\mathbb{K}}(G)$ is defined for all graphs G that are *not* k-colorable, and for all fields $\mathbb{K}$ for which the characteristic does not divide k. Our main result is the following:

THEOREM 2.1. $N_{k,\mathbb{K}}(G) \equiv 1 \pmod{k}$, *for all* $k, \mathbb{K}, G$. *Furthermore* $N_{k,\mathbb{K}}(G) \geq k+1$ *if* $k > 3$.

PROOF. Let $G = (V, E)$, and let $\mathcal{I}_G$ denote the k-coloring ideal of G. Then, $\mathcal{I}_G$ is generated by vertex polynomials $\nu_i = x_i^k - 1$ (for $i \in V(G)$) and edge polynomials $\eta_{ij} = (x_i^k - x_j^k)/(x_i - x_j)$ (for $(i,j) \in E(G)$). We note that $\mathcal{I}_G$ has a Nullstellensatz certificate over $\mathbb{K}[x_1, \ldots, x_n]$ if and only if it has such a certificate over $\mathbb{K}[x_1, \ldots, x_n]/\langle x_1^k - 1, \ldots, x_n^k - 1\rangle$. Therefore, we may consider only the edge polynomials η_{ij} and assume that degrees of variables are taken modulo k, that is, $x_i^k = 1$ for every i.

Suppose that $\{\alpha_{ij}\}$ is a Nullstellensatz certificate of degree d, so that $\sum_{ij \in E} \alpha_{ij}\eta_{ij} = 1$. We write $\alpha_{ij} = \sum_t \alpha_{t,ij}$, where $\alpha_{t,ij}$ is homogeneous of degree t. Equating terms of like degree, we conclude:

$$\sum_{ij \in E, t \equiv 1} \alpha_{t,ij}\eta_{ij} = 1 \quad \text{and}$$

$$\sum_{ij \in E} \alpha_{t,ij}\eta_{ij} = 0, \quad \text{for every } t \not\equiv 1 \pmod{k}$$

Hence, letting $\beta_{ij} = \sum_{t \equiv 1} \alpha_{t,ij}$, observe that $\{\beta_{ij}\}$ is a Nullstellensatz certificate with degree congruent to 1 modulo k. We conclude that $N_{k,\mathbb{K}}(G) \equiv 1 \pmod{k}$.

Now consider $k > 3$ and suppose towards a contradiction that there exists a Nullstellensatz certificate $\{\alpha_{ij}\}$ of degree at most 1. By our logic above, we need only consider terms in α_{ij} for which the degree is 1 modulo k. Suppose therefore that $\alpha_{ij} = \sum_h c_{h,ij} x_h$, so that

$$\sum_{h \in V,\, ij \in E} c_{h,ij} x_h \eta_{ij} = 1.$$

Notice that $c_{h,ij}x_h\eta_{ij}$ can contain a constant term only when h equals i or j, in which case $x_h(x_i^{k-1})$ or $x_h(x_j^{k-1})$ equals 1. We conclude that

$$1 = \sum_{ij \in E} (c_{i,ij} + c_{j,ij}). \quad (1)$$

Observe that $c_{i,ij}x_i\eta_{ij}$ contains a term of the form $c_{i,ij}x_i^{k-2}x_j^2$. Since $k > 3$, the monomial $x_i^{k-2}x_j^2$ occurs for only one other choice of h' and $i'j'$, namely $i' = i$ and $h' = j' = j$. In order for this term to cancel in the final sum, therefore, we must have $c_{j,ij} = -c_{i,ij}$ for all $ij \in E$. However, this contradicts (1). We conclude that for $k > 3$, no Nullstellensatz certificate exists of degree 1, and therefore that $N_{k,\mathbb{K}}(G) \geq k + 1$. $\square$

We observe that Theorem 2.1 is a generalization of Lemmas 4.0.48 and 4.0.49 of [31], which only deals with the graph-3-colorability case.

EXAMPLE 2.2. *Consider the following* ***incomplete*** *degree four certificate for non-3-colorability over $\mathbb{F}_2$. Observe that the coefficient for the vertex polynomial $(x_1^3 + 1)$ contains only monomials of degree zero and degree three, whereas the coefficient for the edge polynomial $(x_1^2 + x_1x_3 + x_3^2)$ contains only monomials of degree one or degree four. This certificate demonstrates the modular degree grouping of the monomials in the certificates, as described by Theorem 2.1. We do not display the full certificate here due to space considerations.*

$$\begin{aligned}
1 = {} & (1 + x_1x_3x_5 + x_1x_3x_7 + x_1x_4x_5 + x_1x_4x_6 + x_1x_5x_6 \\
& + x_1x_5x_7 + x_2^2x_5 + x_2^2x_7 + x_2x_4x_5 + x_2x_4x_6 + x_2x_6x_7 \\
& + x_3x_4x_5 + x_3x_4x_7 + x_4x_6x_7 + x_5x_6x_7)(x_1^3 + 1) \\
& + (x_2 + x_4 + x_5 + x_1^2x_2x_5 + x_1^2x_2x_7 + x_1^2x_3x_5 + x_1^2x_3x_7 \\
& + x_1^2x_4x_5 + x_1^2x_4x_6 + x_1^2x_6x_7 + x_1x_2x_4x_5 + x_1x_2x_4x_7 \\
& + x_1x_3x_4x_5 + x_1x_3x_4x_7 + x_1x_3x_5x_6 + x_1x_3x_5x_7 \\
& + x_1x_3x_6x_7 + x_1x_4x_5x_6 + x_1x_4x_5x_7 + x_1x_4x_6x_7 \\
& + x_1x_5x_6x_7 + x_2x_4x_5x_6 + x_2x_4x_5x_7 + x_2x_5x_6x_7 \\
& + x_3x_4x_5x_6 + x_3x_4x_5x_7)(x_1^2 + x_1x_2 + x_2^2) + \cdots
\end{aligned}$$

2.1.1 Experiments on change of coefficient field

In Table 1, we display experimental data on minimum-degree Nullstellensatz certificates for various cases of graph-k-colorability and various finite fields. This data was found via the high-performance computing cluster at the US Naval Academy (and the NulLa software [15]). Observe that the Nullstellensatz certificate computed by testing the complete graph K_7 for non-6-colorability is **not** the minimum degree seven, instead the minimum-degree certificate is the next higher residue class (degree thirteen). Additionally we performed many more experiments not presented in Table 1. We tested non-3-colorability for K_4 for the first 1,000 prime finite fields. The certificate degree was degree one for finite fields $\mathbb{F}_2$ and $\mathbb{F}_5$, changed to the next highest degree (degree four) at $\mathbb{F}_7$, and then remained degree four for the next 997 primes (up to $\mathbb{F}_{7919}$). We also tested non-4-colorability for K_5 for the first 1,000 primes: the minimum-degree remained five for the entire series of computations. In general, Table 1 suggests that the bound $N_{k,\mathbb{K}}(G) \geq k + 1$ for $k \geq 4$ is not tight for large k. We propose the following conjecture:

CONJECTURE 2.3. *For every field $\mathbb{K}$ and for every positive integer m, there exists a constant k_0 with the following property. For each $k > k_0$ and G a non-k-colorable graph, every Nullstellensatz certificate of the k-coloring ideal of G has degree at least $mk + 1$.*

Graph	k	Theorem 2.1 Possible degrees	$\mathbb{F}_2$	$\mathbb{F}_3$	$\mathbb{F}_5$	$\mathbb{F}_7$
K_4	3	$1, 4, 7, 10, \ldots$	1	–	4	4
K_5	4	$5, 9, 13, \ldots$	–	5	5	5
K_6	5	$6, 11, 16, \ldots$	6	6	–	11
K_7	6	$7, 13, 19, \ldots$	–	–	13	13
K_8	7	$8, 15, 22, \ldots$	8	≥ 15	≥ 15	–
K_9	8	$9, 17, 25, \ldots$	–	≥ 17	≥ 17	≥ 17
K_{10}	9	$10, 19, 28, \ldots$	≥ 19	–	≥ 19	≥ 19
K_{11}	10	$11, 21, 31, \ldots$	–	≥ 21	–	≥ 21

Table 1: The minimum degree of Nullstellensatz certificates for complete graphs over $\mathbb{F}_p$. Note that computations are only possible when k and p are relatively prime (incompatible pairs (k,p) are denoted by $-$).

2.2 The robust Hardness of Colorful Gröbner bases

We know it is NP-hard to compute Gröbner bases. It is even known the problem is EXPSPACE-complete (see [23, 33]), and the maximum degree of the basis can become very large. In [24, 36, 35] the authors presented bounds for the degree of a reduced Gröbner basis for an ideal whose generators have degree bounded by d. E.g., the authors of [35] show that a Gröbner basis of an r-dimensional ideal has degree at most $2\left(\frac{1}{2}d^{n-r} + d\right)^{2^r}$. For the case of general zero-dimensional ideals, this bound reduces to $2\left(\frac{1}{2}d^n + d\right)$. In [39], a lower bound of d^n for zero-dimensional ideals is given by a suitable example. Finally from the work of Lazard and Brownawell [6, 29] it follows an $n(d-1)$ bound on zero-dimensional ideals with generators having no common zeros at infinity. These include our coloring ideals, $\mathcal{I}_k(G)$.

On the other hand, it is well-known that some combinatorial problems are even hard to approximate or it is hard to find partial solutions. Here we discuss how the hardness of finding suboptimal or approximate solutions to graph k-coloring can be translated into similar results for the computation of Gröbner bases, therefore showing some kind of "robust hardness" for Gröbner bases computation. We will use the following theorem.

THEOREM 2.4 (SEE [27]). *It is NP-hard to color a 3-colorable graph with 4 colors. More generally, for every $k \geq 3$ it is NP-hard to color a k-chromatic graph with at most $k + 2\left\lfloor \frac{k}{3} \right\rfloor - 1$ colors.*

We now translate this theorem into a statement about Gröbner bases. Having additional colors to work with allows us to ignore certain vertices of our graph and color these later using our extra colors. Algebraically, this corresponds to ignoring certain variables and computing a Gröbner basis for the partial coloring ideal on the remaining variables.

DEFINITION 2.5. *Given a set of polynomials $\mathcal{F} \subseteq \mathbb{K}[x_1, \ldots, x_n]$, we say that a subset X of the variables $x_1, \ldots, x_n$ is* independent *on $\mathcal{F}$ if no two variables in X appear together in any element of $\mathcal{F}$.*

Clearly independent sets in our coloring ideal generators correspond to independent sets of vertices of the graph.

DEFINITION 2.6. *Define the* strong c-partial Gröbner *problem as follows. Given as input, a set $\mathcal{F}$ of polynomials on a set X of variables, output the following:*

- *disjoint $X_1, \ldots, X_b \subseteq X$, such that $b \leq c$ and each X_i is an independent set of variables,*
- *$X' \subseteq X$, where $X' = X \setminus \left(\bigcup_i X_i\right)$ (i.e., we have taken away at most c independent sets of variables),*
- *$\mathcal{F}' \subseteq \mathcal{F}$ such that $\mathcal{F}'$ consists of all polynomials in $\mathcal{F}$ involving only variables in X',*
- *a Gröbner basis for $\langle \mathcal{F}' \rangle$ over X' (where the monomial order on X is restricted to a monomial order on X').*

THEOREM 2.7. *Suppose that we are working over a polynomial ring $\mathbb{K}[x_1, \ldots, x_n]$ under some elimination order on the variables (such as lexicographic order).*

Let $k \geq 3$ be an integer, and set $c = 2\left\lfloor \frac{k}{3} \right\rfloor - 1$. Unless P = NP, there is no polynomial-time algorithm $\mathcal{A}$ that solves the strong c-partial Gröbner problem (even if we restrict to sets of polynomials of degree at most k).

The following lemma will be useful in our proof.

LEMMA 2.8. *Suppose that we are given a Gröbner basis $\mathcal{G}$ for the k-coloring ideal $\mathcal{I}_G$ of a graph G, with respect to a given elimination order. Assuming the variety $\mathcal{V}(\mathcal{I}_G)$ is non-empty, there is an algorithm that finds some solution $x \in \mathcal{V}(\mathcal{I}_G)$ in time polynomial in the encoding length of $\mathcal{G}$, and therefore identifies a k-coloring of G.*

PROOF. Suppose without loss of generality that our elimination order gives $x_n > x_{n-1} > \cdots > x_1$. We may enumerate the elements of $\mathcal{G}$: $g_1, g_2, \ldots, g_n$, so that g_1 is univariate in x_1, the polynomial g_2 is bivariate (or univariate) in x_1 and x_2, etc. See [10, Chap. 3] for more details.

As always with coloring ideals, we assume that k is not divisible by the characteristic of $\mathbb{K}$. Hence, in a standard result of Galois theory, there exists a primitive kth root of unity over $\mathbb{K}$, and so the kth cyclotomic polynomial Φ_k over $\mathbb{K}$ must be nontrivial. In order to find Φ_k, we repeatedly divide the polynomial $x^k - 1$ by its greatest common factors with polynomials $x^\ell - 1$ for $\ell < k$; these gcd's can be found by the Euclidean algorithm. The time required may be exponential in k but is obviously independent of $\mathcal{G}$.

Note that every proper k-coloring of G must correspond to a solution to all polynomials in $\mathcal{G}$. Thus, if some collection of roots of unity forms a solution, any permutation of those roots of unity must also. Therefore, every root of Φ_k must also be a root of g_1, that is, Φ_k divides g_1.

The Elimination Theorem [10, Chap. 3] now guarantees that any solution to g_1 and g_2 extends to a solution for all $\mathcal{G}$. For each i with $1 \leq i \leq k$, we set $x_1 = \omega$ and $x_2 = \omega^i$. We then test, using the Euclidean algorithm, whether there exists a common solution to $g_1(x_1)$, $g_2(x_1, x_2)$, and $\Phi_k(x_1)$, considered as polynomials in ω. If so, then every primitive root ω gives us a common solution of g_1, g_2.

Having identified i, set $i_2 = i$ and $i_1 = 1$, so that $x_1 = \omega^{i_1}$, $x_2 = \omega^{i_2}$ is a partial solution. We proceed as above to check each i_3 satisfying $1 \leq i_3 \leq k$. Using the Euclidean algorithm, we test if there is a common solution ω to $g_1(x_1)$, $g_2(x_1, x_2)$, $g_3(x_1, x_2, x_3)$, $\Phi_k(x_1)$, if we set $x_1 = \omega^{i_1}$, $x_2 = \omega^{i_2}$, $x_3 = \omega^{i_3}$. This enables us to find a partial solution for x_1, x_2, x_3. Continuing in this fashion, we find a complete solution for all variables. $\square$

PROOF OF THEOREM 2.7. The proof is by contradiction. Let $G = (V, E)$ be a k-colorable graph and assume such a polynomial-time algorithm $\mathcal{A}$ exists. We will give a method for producing a proper $(k + c)$-coloring of G. This contradicts Theorem 2.4 under the assumption that P $\neq$ NP, as mentioned above.

Let us apply the algorithm $\mathcal{A}$ to our coloring polynomials $\mathcal{F}_G$ for the graph G, giving us a Gröbner basis $\mathcal{G}$. Note that the input consists of $|V| + |E|$ polynomials with degree $\leq k$ and length $\leq k$. Thus, $\mathcal{F}_G$ has polynomial size in k and the encoding length of G, and by assumption $\mathcal{A}$ terminates in time which is polynomial in both of these quantities.

Observe that the variables in $\mathcal{F}_G$ correspond to vertices of G, and an independent set of variables corresponds to an independent set of vertices. Assume that the independent sets of variables which were ignored by $\mathcal{A}$ are $X_1, X_2, \ldots, X_b$ for $b \leq c$. Let $I_1, I_2, \ldots, I_b$ be the corresponding independent sets of vertices. The Gröbner basis $\mathcal{G}$ corresponds to proper k-colorings of $G' = G \setminus (\cup_i I_i)$. Therefore, Lemma 2.8 implies that we can identify some proper coloring of G' using the colors $1, \ldots, k$. Note that in order to apply this lemma, we must be working with an elimination order over our restricted polynomial ring; this is true since the restriction of an elimination order to a smaller set of variables is also an elimination order.

Now color the independent sets $I_1, \ldots, I_b$ in the colors $k+1, \ldots, k+b$. This gives us a proper coloring of G using at most

$$k + b \leq k + c = k + 2\left\lfloor \frac{k}{3} \right\rfloor - 1$$

colors. By Theorem 2.4, this is impossible to construct in polynomial time, giving us a contradiction, as desired. $\square$

Theorem 2.7 demonstrates how results on coloring graphs translate effectively to results on Gröbner bases. For reference, a weaker result may be proven without recourse to the full power of the coloring ideal.

DEFINITION 2.9. *Define the* weak c-partial Gröbner *problem as follows. Given, as input, a set $\mathcal{F}$ of polynomials on a set X of variables, output the following:*

- *$X' \subseteq X$ such that $|X'| \geq |X| - c$,*
- *$\mathcal{F}' \subseteq \mathcal{F}$ such that $\mathcal{F}'$ consists of all polynomials in $\mathcal{F}$ involving only variables in X',*
- *a Gröbner basis for $\langle \mathcal{F}' \rangle$ over X' (where the monomial order on X is restricted to a monomial order on X').*

THEOREM 2.10. *For constant c, there is no polynomial-time algorithm to solve the weak c-partial Gröbner problem, unless P=NP. This holds even if we restrict to sets of polynomials of degree at most 3.*

Our proof will use the following lemma.

LEMMA 2.11. *Suppose that $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_m$ are sets of polynomials on disjoint sets of variables (that is, no variable appears both in a polynomial of $\mathcal{F}_i$ and in a polynomial of $\mathcal{F}_j$). Then, the reduced Gröbner basis of $\langle \cup_i \mathcal{F}_i \rangle$ is the union of the reduced Gröbner bases for the individual $\langle \mathcal{F}_i \rangle$.*

PROOF. Let $\mathcal{G}_i$ be the reduced Gröbner bases for the $\langle \mathcal{F}_i \rangle$, respectively, and set $\mathcal{G} := \cup_i \mathcal{G}_i$. For a set S of polynomials, we use $\mathcal{L}(S)$ to denote the ideal generated by the leading terms of elements of S.

Note first that every leading term of $\langle \cup_i \mathcal{F}_i \rangle$ is also a leading term of some $\langle \mathcal{F}_i \rangle$ and hence is contained in $\mathcal{L}(\mathcal{G})$. Conversely, every leading term of $\mathcal{G}$ is also a leading term of $\langle \mathcal{F}_i \rangle$, for some i, and therefore is contained in $\langle \cup_i \mathcal{F}_i \rangle$. We conclude that

$$\langle \cup_i \mathcal{F}_i \rangle = \mathcal{L}(\mathcal{G}),$$

and therefore $\mathcal{G}$ is a Gröbner basis of $\cup_i \langle \mathcal{F}_i \rangle$. $\square$

PROOF OF THEOREM 2.10. Suppose that there exists an algorithm $\mathcal{A}$ for c-partial Gröbner that runs in time at most $p(s)$, where s is the size of the input. Let $\mathcal{F}$ be a system of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ with input size s, such that the degree of every polynomial in $\mathcal{F}$ is at most 3. We show how to use $\mathcal{A}$ to compute a Gröbner basis for $\langle \mathcal{F} \rangle$ in polynomial time, which will lead to a contradiction.

Construct copies $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{c+1}$ of $\mathcal{F}$ on disjoint sets of variables, so that $\mathcal{F}_i$ includes polynomials over the variables $x_{i,1}, x_{i,2}, \ldots, x_{i,n}$. The size of $\cup_i \mathcal{F}_i$ is obviously $(c+1)s$. Now run $\mathcal{A}$ on $\cup_i \mathcal{F}_i$, removing at most c variables from $\cup_i \mathcal{F}_i$. In the process, we remove certain polynomials from $\mathcal{F}_i$ to yield sets $\mathcal{F}'_i$ of polynomials. The output of $\mathcal{A}$ is a Gröbner basis $\mathcal{G}$ for $\langle \cup_i \mathcal{F}'_i \rangle$.

Now, since there are $c+1$ disjoint sets of variables $\{x_{i,1}, x_{i,2}, \ldots, x_{i,n}\}$, there must exist at least one i such that we have not removed any variable in $\{x_{i,1}, x_{i,2}, \ldots, x_{i,n}\}$. For this value of i, we have $\mathcal{F}'_i = \mathcal{F}_i$. Transforming $\mathcal{G}$ to a reduced Gröbner basis is routine and can be performed in polynomial time. Applying Lemma 2.11, we see that the restriction of $\mathcal{G}$ to $\{x_{i,1}, x_{i,2}, \ldots, x_{i,n}\}$ gives a reduced Gröbner basis for $\mathcal{F}'_i = \mathcal{F}_i$. This immediately gives us a reduced Gröbner basis for $\langle \mathcal{F} \rangle$.

Observe that $(c+1)s$ is the size of our input $\cup_i \mathcal{F}_i$ to $\mathcal{A}$. Therefore, the time required by our algorithm is at most $p((c+1)s) \leq (c+1)^{\deg(p)} p(s)$. Since $\mathcal{F}$ was chosen arbitrarily, this implies that for every family of polynomials of input size s, a Gröbner basis can be found in polynomial time at most $(c+1)^{\deg(p)} p(s)$. However, since 3-coloring is NP-hard, the general problem of finding a Gröbner basis cannot be performed in polynomial time, even if we assume that every $f \in \mathcal{F}$ has degree at most 3. Thus we have a contradiction, and conclude that the algorithm $\mathcal{A}$ cannot exist. $\square$

Comparing Theorems 2.7 and 2.10, we see that the latter allows us to remove only a constant number of individual variables, not a constant number of independent sets. Furthermore, the set of polynomials constructed in Theorem 2.10 is *disconnected*, according to the following Definition 2.12, while the set of polynomials constructed in Theorem 2.7 is *connected*. It appears more natural to consider connected sets of polynomials, which occur in many applications.

DEFINITION 2.12. *We say that a set $\mathcal{F}$ of polynomials is* disconnected *if we can partition $\mathcal{F}$ into $\mathcal{F}_1, \mathcal{F}_2$ such that the variables for $\mathcal{F}_1$ and $\mathcal{F}_2$ are disjoint. Otherwise, we say that $\mathcal{F}$ is* connected.

3. GRÖBNER BASES FOR CHORDAL GRAPHS

Even though graph coloring is hard for general graphs, the problem can be solved in linear time for chordal graphs (see e.g. [25]). Taking advantage of the structure of chordal graphs, we develop a polynomial time algorithm that computes a Gröbner basis for the k-coloring ideal $\mathcal{I}_G$ of a given chordal graph G. The monomial order we consider is related to the perfect elimination ordering of G.

We begin with some basic definitions in graph theory (see also e.g. [18].) Recall that a graph $G = (V, E)$ is *chordal* if every cycle of length more than 3 has a chord, or equivalently, every induced cycle in the graph has length 3. A vertex $v \in V$ is *simplicial* if its neighbors form a clique. A graph is *recursively simplicial* if it contains a simplicial vertex v such that the induced graph $G[V \setminus \{v\}]$ produced by removing v and its incident edges, is recursively simplicial[1]. If G is recursively simplicial, there exists an ordering on V, called a *perfect elimination ordering*, such that when the vertices of G are removed in that order, each vertex will be simplicial at the time of removal.

PROPOSITION 3.1 ([21]). *Let $G = (V, E)$ be a graph. Then G is chordal if and only if it is recursively simplicial.*

For a vertex $v \in V$, the neighborhood of v in G is the set $\mathcal{N}(v) = \{w \in V : (v, w) \in E(G)\}$. If $U \subseteq V$ is a subset of the vertices forming a clique in G, then we define

$$G^{+U} := \big(V \cup \{n+1\}, E \cup \{(j, n+1) : j \in U\}\big)$$

to be the graph obtained by adding a new vertex and connecting it to all $u \in U$. Note that this operation is the inverse of deleting a simplicial vertex of G, and, by Proposition 3.1, every chordal graph can be constructed in this way.

Algorithm 1 constructs a Gröbner basis $\mathcal{G}$ for a chordal graph G, building up the graph one vertex at a time according to the reverse elimination order. Each newly added vertex adds a polynomial to $\mathcal{G}$. At any point, having constructed the graph $G' \subseteq G$, the set of polynomials added will form a Gröbner basis for the coloring ideal of G'.

3.1 Preliminaries

Recall the following definitions. The *kth elementary symmetric polynomial* $\sigma_k(x_1, \ldots, x_n)$ over n variables is

$$\sigma_k(x_1, \ldots, x_n) := \sum_{1 \leq j_1 < \cdots < j_k \leq n} x_{j_1} \cdots x_{j_k} \quad .$$

The *kth complete homogeneous symmetric polynomial* $S_k(x_1, \ldots, x_n)$ over n variables is given by

$$S_k(x_1, \ldots, x_n) := \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} x_{j_1} \cdots x_{j_k} \quad .$$

Note that both polynomials are degree-k-homogeneous, but the monomials of σ_k are by definition square-free, while S_k can contain higher powers of a variable.

LEMMA 3.2. *For a positive integer k, let $\zeta_1, \zeta_2, \ldots, \zeta_k$ be the kth roots of unity in some order. Then, for every $k > r$, $S_{k-r}(\zeta_1, \zeta_2, \ldots, \zeta_r, x) = (x - \zeta_{r+1})(x - \zeta_{r+2}) \cdots (x - \zeta_k)$.*

PROOF. It suffices to prove

$$S_{k-r}(\zeta_1, \zeta_2, \ldots, \zeta_r, x) \cdot (x - \zeta_1) \cdots (x - \zeta_r) = x^k - 1 \quad .$$

[1] The graph on zero vertices is recursively simplicial.

Consider the degree d-homogeneous polynomial $\sigma_i(\zeta_1,\dots,\zeta_r)S_{d-i}(\zeta_1,\dots,\zeta_r)$. For every monomial x^α with $|\alpha| = d$ and $\mathrm{supp}(\alpha) = m$ (the number of non-zero elements in α equals m), its coefficient is the number of square-free factors of degree i, that is, $\binom{m}{i}$. Summing up these coefficients over d with alternating signs gives that the coefficient of x^α in

$$\sum_{i=0}^{d}(-1)^{d-i}\sigma_i(\zeta_1,\dots,\zeta_r)S_{d-i}(\zeta_1,\dots,\zeta_r)$$

equals

$$\sum_{i=0}^{m}(-1)^{d-i}\binom{m}{i} = 0 \quad .$$

Therefore,

$$\sum_{i=0}^{d}(-1)^{d-i}\sigma_i(\zeta_1,\dots,\zeta_r)S_{d-i}(\zeta_1,\dots,\zeta_r) = 0,$$

for every $d \in \{0,\dots,k-1\}$. Now, since $\zeta_1,\dots,\zeta_k$ are the roots of unity, we know that, for every $d \in \{1,\dots,k-1\}$:

$$\sum_{i=0}^{d}\sigma_i(\zeta_1,\dots,\zeta_r)\sigma_{d-i}(\zeta_{r+1},\dots,\zeta_k) = \sigma_d(\zeta_1,\dots,\zeta_k) = 0.$$

We now have identical recursions for $S_d(\zeta_1,\dots,\zeta_r)$ and $(-1)^d\sigma_d(\zeta_{r+1},\dots,\zeta_k)$. In the base case, $S_0(\zeta_1,\dots,\zeta_r) = 1 = (-1)^0\sigma_0(\zeta_{r+1},\dots,\zeta_k)$. We conclude that for all d,

$$S_d(\zeta_1,\dots,\zeta_r) = (-1)^d\sigma_d(\zeta_{r+1},\dots,\zeta_k).$$

Therefore,

$$\begin{aligned}
&S_{k-r}(\zeta_1,\dots,\zeta_r,x)\cdot\prod_{i=1}^{r}(x-\zeta_i)\\
&=\sum_{d=0}^{k-r}S_d(\zeta_1,\dots,\zeta_r)x^{k-r-d}\cdot\prod_{i=1}^{r}(x-\zeta_i)\\
&=\sum_{d=0}^{k-r}(-1)^d\sigma_d(\zeta_{r+1},\dots,\zeta_k)x^{k-r-d}\cdot\prod_{i=1}^{r}(x-\zeta_i)\\
&=\prod_{i=r+1}^{k}(x-\zeta_i)\cdot\prod_{i=1}^{r}(x-\zeta_i) = x^k-1.
\end{aligned}$$

□

3.2 The Algorithm

BuildGröbnerBasis (Algorithm 1) successively tests vertices of a chordal graph G for simpliciality and obtains a perfect elimination order while concurrently adding new polynomials to a set $\mathcal{G}$. At termination, $\mathcal{G}$ is a Gröbner basis for $\mathcal{I}_G$ with respect to the lexicographic order with variables ordered according to a perfect elimination order of G.[2]

For a clique $U = \{u_1, u_2, \dots, u_r\}$ and vertex v in our graph, we will use the notation $S_{k-r}(U, v)$ to denote the polynomial $S_{k-r}(x_{u_1}, x_{u_2}, \dots, x_{u_r}, x_v)$.

As we have seen above, exactly one polynomial is added to $\mathcal{G}$ for every vertex of G. From the definition of $S_k(x_1,\dots,x_n)$, we see that its length is $\binom{k+n-1}{n-1}$ and its degree is k. So the polynomials S_i added to $\mathcal{G}$ have length $\binom{k}{|U_i|}$ and degree $(k-|U_i|)$. Both quantities are polynomial in k and constant for a constant number k of colors.

[2]The existence of this algorithm was first conjectured by experimental work of Pernpeintner [38].

Algorithm 1 Produces a Gröbner basis for the k-coloring ideal $\mathcal{I}_G$ for G chordal

```
function BuildGröbnerBasis(G. k)
Input: A chordal graph G, integer k
Output: A Gröbner basis for I_G
    G_n ← G
    𝒢 ← ∅
    for all i ∈ {n − 1, . . . , 1} do
        for all v ∈ V_{i+1} do
            if IsSimplicial(v, G_{i+1}) then
                v_i ← v
                U_i ← N(v)
                if |U_i| ≥ k then
                    return {1}
                end if
                G_i ← G_{i+1} − v
                𝒢 ← 𝒢 ∪ {S_{k−|U_i|}(U_i, v_i)}
            end if
        end for
    end for
    return 𝒢
end function

function IsSimplicial(v, G)
Input: A vertex v of the graph G and the graph G itself
Output: True if v is simplicial in G; False otherwise
    d ← deg(v)
    for all w ∈ N(v) do
        if |N(v) ∩ N(w)| < d − 1 then
            return false
        end if
    end for
    return true
end function
```

Finally let us discuss an important aspect of our algorithm. If G is not k-colorable, then in the process of **BuildGröbnerBasis** we would intuitively expect the constant polynomial 1 to appear somewhere in the set $\mathcal{G}$. This can be shown formally: Assume that $\chi(G) = \chi > k$, and we try to find a Gröbner basis for the k-coloring ideal of G. Since G is chordal, it is also perfect, and thus has a χ-clique $\{v_1,\dots,v_\chi\}$. We assume without loss of generality that these vertices are ordered ascendingly with respect to the perfect elimination order from the algorithm.

In the step, where v_{k+1} is removed from the graph, we have $\{v_1,\dots,v_k\} \subseteq \mathcal{N}(v_{k+1})$, and therefore, we add the complete polynomial of degree 0

$$S_{k-k}(x_{v_1},\dots,x_{v_k},x_{v_{k+1}}) = 1 \quad .$$

Hence, BuildGröbnerBasis detects non-k-colorability on the fly. This observation allows us to do the following: If we find a simplicial vertex of degree $\geq k$, then we can stop immediately and return the trivial Gröbner basis $\mathcal{G} = \{1\}$. On the other hand, we can be sure that if there is no such forbidden vertex, then G is k-colorable.

3.3 Correctness

Lemma 3.3 (Lemma 2.2 from [26]). *Let G be a graph. Then $\mathcal{I}_G$ is a radical ideal.*

Proposition 3.4 ([10] 2.9 Proposition 4). *Let $P \subset \mathbb{K}[x_1, \ldots, x_n]$ be a finite set, and let $p_1, p_2 \in P$ be such that*

$$lcm(LM(p_1), LM(p_2)) = LM(p_1) \cdot LM(p_2) \quad ,$$

where LM denotes the leading monomial of a polynomial. Then,

$$S\text{-}pair(p_1, p_2) \rightarrow_P 0.$$

Recall that $v_i \in \mathcal{I}_G$, and $\eta_{ij} \in \mathcal{I}_G$ are the vertex and edge polynomials, respectively.

Lemma 3.5. *Let G be a chordal graph on n vertices, and let $\succ$ be a term order. Let $U = \{u_1, \ldots, u_r\}$ be an r-clique in G, and choose a Gröbner basis $\mathcal{G}$ of $\mathcal{I}_G$. Set $p = S_{k-r}(x_{u_1}, \ldots, x_{u_r}, x_{n+1})$. Then,*

$$\langle \mathcal{G}, p \rangle = \langle \mathcal{G}, \nu_{n+1}, \eta_{u_1,n+1}, \ldots, \eta_{u_r,n+1} \rangle = \mathcal{I}_{G+U} \quad .$$

Proof. We show that $\langle \mathcal{G}, p \rangle$ is a radical ideal, and that both ideals generate the same variety. Then the claim follows from the bijection between varieties and radical ideals [10, §4.2, Theorem 7].

Consider some setting of the variables $x_{u_1}, \ldots, x_{u_r}$ to distinct kth roots of unity $\zeta_1, \ldots, \zeta_r$, and suppose that $\zeta_{r+1}, \ldots, \zeta_k$ are the other kth roots of unity, in some order. By Lemma 3.2, we have $p = \prod_{i=r+1}^{k}(x_{n+1} - \zeta_i)$. This implies that $p(x_{u_1}, x_{u_2}, \ldots, x_{u_r}, x_{n+1})$ is a square-free polynomial so $\langle p \rangle$ is a radical ideal. The ideal $\langle \mathcal{G} \rangle$ is also radical, since it is the coloring ideal of a graph (Lemma 3.3). But then

$$\begin{aligned} \text{rad}(\langle \mathcal{G}, p \rangle) &= \text{rad}(\langle \mathcal{G} \rangle \cap \langle p \rangle) = \text{rad}(\langle \mathcal{G} \rangle) \cap \text{rad}(\langle p \rangle) \\ &= \langle \mathcal{G} \rangle \cap \langle p \rangle = \langle \mathcal{G}, p \rangle \end{aligned}$$

as claimed. The second equality is [10, §4.3, Proposition 16],

Now consider $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathcal{V}(\langle \mathcal{G}, p \rangle)$. Since $u_1, \ldots, u_r$ form a clique, we know that x_{u_i} are distinct kth roots of unity. Then, by Lemma 3.2, x_{n+1} is a kth root of unity, and so $\nu_{n+1} = 0$. Moreover, $x_{n+1} \neq x_{u_i} \ \forall \ i \in \{1, \ldots, r\}$, which implies that $\eta_{u_i,n+1} = 0$. We conclude that $\mathbf{x} \in \mathcal{V}(\mathcal{I}_{G+U})$.

Conversely, consider $\mathbf{x} = (x_1, \ldots, x_{n+1}) \in \mathcal{V}(\mathcal{I}_{G+U})$. The generator polynomials $\nu_1, \ldots, \nu_r, \nu_{n+1}$ and $\eta_{u_1,n+1}, \ldots, \eta_{u_r,n+1}$ ensure that $x_{u_1}, \ldots, x_{u_r}, x_{n+1}$ are distinct kth roots of unity. Hence $p(\mathbf{x}) = 0$ and $\mathbf{x} \in \mathcal{V}(\langle \mathcal{G}, p \rangle)$. $\square$

Lemma 3.6. *Given a chordal graph G, every Gröbner basis $\mathcal{G}$ of $\mathcal{I}_G$ with respect to $\succ$, $\mathcal{G} \cup \{p\}$ is a Gröbner basis of $\mathcal{I}_{G+U}$ with respect to an extended term order $\succ'$, where p is again defined as in Lemma 3.5.*

Proof. Lemma 3.5 shows that $\langle \mathcal{G}, p \rangle = \mathcal{I}_{G+U}$. Hence, it is left to show that all S-polynomials in $\mathcal{G} \cup \{p\}$ reduce to 0. We only have to consider S-pairs that involve the new polynomial p.
By definition of $\succ'$, we have that $\text{LM}_{\succ'}(p) = x_{n+1}^{k-r}$, which is relatively prime to all $g \in \mathcal{G}$, since x_{n+1} does not appear in this basis. Therefore, for all $g \in \mathcal{G}$, S-pair$(g, p) \rightarrow_{\mathcal{G} \cup \{p\}} 0$ by Proposition 3.4. This is sufficient for $\mathcal{G}' := \mathcal{G} \cup \{p\}$ to be a Gröbner Basis. $\square$

Theorem 3.7. *For G chordal, the set $\mathcal{G}$ output by* BuildGröbnerBasis(G) *is a Gröbner basis for $\mathcal{I}_G$ under the Lex order, with variables ordered according to the perfect elimination order of G established in the algorithm.*

Proof. Note that $\{p_1 := \nu_n\}$ is a Gröbner basis for G_1. By Lemma 3.6, this basis can be extended in $n-1$ steps by adding p_i as constructed in the algorithm. Therefore, $\mathcal{G} = \{p_1, \ldots, p_n\}$ is a Gröbner basis of $G_n = G$ with respect to the extended vertex order, which concludes the proof. $\square$

3.4 Complexity

The function IsSimplicial consists of an outer loop with exactly n iterations, each of which calculates the intersection of two subsets of V. This intersection is computed in time $O(n)$ and, hence, the function runs in time $O(n^2)$.

In the main function BuildGröbnerBasis, the two nested **for**-loops are traversed $O(n)$ times each, and every time IsSimplicial is called. The main part of the **if**-case is the assignment of $\mathcal{G}$. If $r = |U_i|$, then building the polynomial $S_{k-|U_i|}(U_i, v_i)$ takes $(k-r) \cdot \binom{k}{r}$ steps, which is clearly in $O(kn^k)$. The remaining statements can be neglected, since they have running time $O(n^2)$. Finally, putting the pieces together, we obtain a total running time of $O(kn^{k+2})$, which is polynomial in n for fixed k.

It is evident that our implementation is not optimal with respect to running time. For instance, finding a simplicial vertex can be done in linear time [40], giving a linear-time procedure that establishes a perfect elimination order on G. Nevertheless, our algorithm shows that finding the Gröbner basis for a chordal graph is polynomial-time solvable, and it describes explicitly the structure of this basis.

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