Topology. SM464. HW 5.

Due: Friday, October/17.

Subject: Complete Metric Spaces, The Banach Fixed-point theorem. Continuous functions.

(1) Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying \( f(x+y) = f(x) + f(y) \) for all \( x, y \in \mathbb{R} \). Prove that there exists \( \alpha \in \mathbb{R} \) such that \( f(x) = \alpha x \) for all \( x \in \mathbb{R} \). \textit{Hint:} Set \( \alpha = f(1) \). Prove that \( f(x \cdot 1) = xf(1) = \alpha x \) when \( x \in \mathbb{Z} \). Then generalize the equation to the case of rational \( x \). Use the fact that the set of rational numbers is dense in the reals.

(2) Section 12. Problem 4.

(3) Use the Banach fixed point theorem to approximate the solution of \( \cos(x) = x \) up to four decimal places. \textit{Hint:} Set \( X = [0,1] \). Show that \( f(x) = \cos(x) \) is a contraction mapping on \( X \). Thus, it must have a fixed point.

(4) Let \( X = [1/2, 1] \cup [1/4, 1/3] \cup [1/6, 1/5] \cup \cdots \cup \{0\} \) and \( d(x, y) = |x - y| \). Prove that \((X, d)\) is a complete metric space. \textit{Hint:} Use the fact that \( X \subset \mathbb{R} \).