Topology. SM464. HW 5.

Due: Friday, October/31.

Subject: Topological Spaces.

(1) Let $X = \mathbb{R}$. Let $\mathcal{F}$ be the family of sets $U \subset \mathbb{R}$ such that for every $x \in U$ there exists an interval $[a, b)$ such that $x \in [a, b) \subset U$. (a) Show that $\mathcal{F}$ is a topology. \textit{Hint: Check the axioms. You can take for granted that $\emptyset \in \mathcal{F}$.} (b) Show that $\mathcal{F}$ is stronger than the standard topology (generated by open intervals) on $\mathbb{R}$. \textit{Hint: Show that if $U$ is a set open in the standard topology, then $U \in \mathcal{F}$.}

(2) Let $X, Y, Z$ be topological spaces. Suppose that $X \cong Y$ (being homeomorphic) and $Y \cong Z$. Prove that $X \cong Z$. \textit{Hint: let $f : X \to Y$ and $g : Y \to Z$ be homeomorphisms. Show that their composition is a homeomorphism between $X$ and $Z$.}

(3) Prove that $(-1, 1)$ is homeomorphic to $\mathbb{R}$. Use $f(x) = \arctan(\pi x/2)$.

(4) Let $X = \{a, b, c, d\}$. Describe the topology generated by the family $B = \{\{a\}, \{c, d\}, \{d\}\}$. 
(5) Let \( X = \mathbb{R} \) be endowed with the standard topology. Give two equivalent definitions/statements for each of the following:

(5-a) \( U \subset \mathbb{R} \) is an open set;
(5-b) \( U \subset \mathbb{R} \) is not an open set.

(6) Let \( X = \mathbb{N} \). Set \( B = \{\{1\}, \{1,3\}, \{1,3,5\}, \{1,3,5,7\}, \ldots\} \). Describe the smallest topology containing the family \( B \). Hint: Consider finite intersections and arbitrary unions of elements of \( B \), add \( \emptyset \) and \( X \).