NO collaboration is permitted on the exam.

(1) Let \((X, d)\) and \((Y, \rho)\) be metric spaces. We say that \(f : X \to Y\) is continuous on \(X\) if \(f\) is continuous at each point \(x \in X\). Prove that if for any open set \(V \subset Y\) its pre-image \(f^{-1}(V)\) is open in \(X\), then \(f\) is continuous on \(X\).

(2) Using the \(\varepsilon - \delta\) definition of continuity, prove that \(f(x) = 1/(1 + x^2)\) is continuous on \(\mathbb{R}\).

(3) Let \((X, d)\) be a metric space and \(A \subset X\). The set \(A\) is called dense in \(X\) if \(\overline{A} = X\). Prove that if \(A\) is dense in \(X\), then for any non-empty open set \(U \subset X\), \(U \cap A \neq \emptyset\). \textit{Hint:} You can prove it by contradiction.

(4) Let \(X\) be a metric space with metric \(d\). Show that \(d_1\), defined by \(d_1(x, y) = d(x, y)/(1 + d(x, y))\), is also a metric on \(X\). (b) Show that the metrics \(d\) and \(d_1\) have the same open spheres with one exception. What is this exception?