Second-Derivative Test for Functions of Two Variables

1 Quadratic Approximation

Suppose we want to approximate the function \(f(x, y)\) to second order around a point \((x_0, y_0)\). That is, we want a polynomial approximation of the form

\[
f(x, y) \approx A + B(x - x_0) + C(y - y_0) + D(x - x_0)^2 + E(x - x_0)(y - y_0) + F(y - y_0)^2 \tag{1}
\]

for constants \(A, B, C, D, E\) and \(F\). For this to agree with our linear (tangent plane) approximation

\[
f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)
\]

we must have \(A = f(x_0, y_0), B = f_x(x_0, y_0)\) and \(C = f_y(x_0, y_0)\).

Now differentiate (1) with respect to \(x\) to get

\[
f_x(x, y) \approx B + 2D(x - x_0) + E(y - y_0).
\]

Since the linear approximation applied to \(f_x\) is

\[
f_x(x, y) \approx f_x(x_0, y_0) + f_{xx}(x_0, y_0)(x - x_0) + f_{xy}(x_0, y_0)(y - y_0),
\]

we must take \(D = \frac{1}{2}f_{xx}(x_0, y_0)\) and \(E = f_{xy}(x_0, y_0)\). Finally, by considering the linear approximation to \(f_y\) we get \(F = \frac{1}{2}f_{yy}(x_0, y_0)\). Thus, the quadratic approximation (1) is

\[
f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \frac{1}{2}f_{xx}(x_0, y_0)(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{1}{2}f_{yy}(x_0, y_0)(y - y_0)^2.
\]
2 Behavior of $f$ at a Critical Point

Suppose now that $(x_0, y_0)$ is a critical point of $f$, so that

$$f_x(x_0, y_0) = f_y(x_0, y_0) = 0.$$  

Then our quadratic approximation becomes

$$f(x, y) - f(x_0, y_0) \approx \frac{1}{2} [f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2],$$

so we need to concentrate on the behavior of

$$f_{xx}(x_0, y_0)(x - x_0)^2 + 2f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + f_{yy}(x_0, y_0)(y - y_0)^2 =$$

$$(x - x_0)^2 \left[ f_{xx}(x_0, y_0) + 2f_{xy}(x_0, y_0)(x - x_0) + f_{yy}(x_0, y_0)(y - y_0) \right]$$

for $(x, y)$ near $(x_0, y_0)$. Think of the expression in braces in (2) as a quadratic function of $t = (x - x_0)/(y - y_0)$:

$$q(t) = f_{xx}(x_0, y_0) + 2f_{xy}(x_0, y_0)t + f_{yy}(x_0, y_0)t^2.$$  

Now $q(t)$ will never change sign as $t$ varies if the roots of $q(t) = 0$ are complex. From the quadratic formula, this happens if

$$4f_{xy}(x_0, y_0)^2 - 4f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) < 0,$$

which is to say

$$f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0.$$  

In this case, $q(t)$ (and thus (2)) will be always positive if its constant term $f_{xx}(x_0, y_0) > 0$, and always negative if $f_{xx}(x_0, y_0) < 0$. So we have the conclusion:

**If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 > 0$, then $f$ has a local minimum at $(x_0, y_0)$ if $f_{xx}(x_0, y_0) > 0$, and a local maximum at $(x_0, y_0)$ if $f_{xx}(x_0, y_0) < 0$.**

Now suppose

$$f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 < 0.$$  

Then the $q(t) = 0$ has two real roots, and so $q(t)$ changes sign. Thus, the expression (2) can be either positive or negative for $(x, y)$ near $(x_0, y_0)$, which means $(x_0, y_0)$ is a saddle point of $f$:

**If $f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2 < 0$, then $f$ has a saddle point at $p$.**