SOME HOPF ALGEBRAS
OF
PHYSICAL INTEREST

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GRADED CONNECTED HOPF ALGEBRAS

\[ A = \bigoplus_{n \geq 0} A_n \]

GRADED ALGEBRA OVER FIELD \( k \) (char. 0); \( A_0 = k1. \)

COPRODUCT \( \Delta: A \to A \otimes A \) RESPECTS GRADING, IS ALGEBRA MAP, AND HAS
\[
\Delta(x) = x \otimes 1 + \sum x' \otimes x'' + 1 \otimes x, \quad 1x'1x'' > 0
\]

\( x \) IS PRIMITIVE IF \( \Delta(x) = 1 \otimes x + x \otimes 1 \)

EXISTENCE OF ANTIPODE \( S: A \to A \)

IS AUTOMATIC; HAVE \( S(1) = 1 \) AND
\[
S(x) = -\sum S(x')x'' - x
\]

FOR \( 1x1 > 0 \). IF \( A \) IS COMMUTATIVE OR COCOMMUTATIVE, \( S^2 = 1d. \) IN GENERAL \( S \) IS ALGEBRA ANTI-AUTOMORPHISM.
DUALS

The graded dual of $A$ is also a Hopf algebra, with product
\[ \langle m^*(uv), w \rangle = \langle uv, \Delta(w) \rangle \]
and co-product
\[ \langle \Delta^*(u), w_1 \otimes w_2 \rangle = \langle u, w_1 w_2 \rangle \]

$A$ is self-dual ($A^* \cong A$) if it admits an inner product $\langle , \rangle$ with
\[ \langle uv, w \rangle = \langle uv, \Delta(w) \rangle. \]

Here is a table of Hopf algebras we will discuss:

<table>
<thead>
<tr>
<th></th>
<th>Commutative?</th>
<th>Co-commut.?</th>
<th>Dual</th>
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<tbody>
<tr>
<td>Sym</td>
<td>Yes</td>
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<td>Sym</td>
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<tr>
<td>QSym</td>
<td>Yes</td>
<td>NO</td>
<td>NSym</td>
</tr>
<tr>
<td>$T$</td>
<td>NO</td>
<td>Yes</td>
<td>$H_k$</td>
</tr>
<tr>
<td>$\beta \simeq H_F$</td>
<td>NO</td>
<td>NO</td>
<td>$H_F$</td>
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Sym AND QSym

Let $B \subset k[[x_1, x_2, \ldots]]$ be the formal power series of bounded degree. ($B$ graded by $|x_i| = 1 \forall i$.)

A series $p \in B$ is in Sym if the coefficient of $x_{i_1} x_{i_2} \cdots x_{i_k}$ agrees with that of $x_{j_1} x_{j_2} \cdots x_{j_k}$ for any two sets $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ of distinct subscripts.

A series $p \in B$ is in $QSym$ if the coefficient of $x_{i_1} x_{i_2} \cdots x_{i_k}$ agrees with that of $x_{j_1} x_{j_2} \cdots x_{j_k}$ for $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

The former condition is more restrictive, so $\text{Sym} \subset Q\text{Sym}$.
Sym as Hopf Algebra

Sym has the basis

\[ m_\lambda = \sum x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k} \] (monomial s.f.'s)

where \( \lambda = \lambda_1 \lambda_2 \lambda_3 \cdots \) is any integer partition

As an algebra,

\[ \text{Sym} = k[e_1, e_2, \cdots] \]

where \( e_i \) is the elementary s.f.

\[ e_i = \frac{m_i^{\lambda_1} \cdots m_i^{\lambda_k}}{\lambda_1! \cdots \lambda_k!} \]

so we also have the basis

\[ e_\lambda = e_1^{\lambda_1} e_2^{\lambda_2} \cdots e_k^{\lambda_k} \]

We can make Sym a Hopf algebra by making the \( e_i \) divided powers:

\[ \Delta(e_\lambda) = \sum_{i=0}^{\infty} e_i \otimes e_{\lambda-i} \]

In general

\[ \Delta(m_\lambda) = \sum_{\nu \vdash \lambda} m_\nu \otimes m_\beta \]
IN FACT, \( \text{Sym} \) IS SELF-DUAL SINCE

THERE IS AN INNER PRODUCT \((\cdot, \cdot)\) WITH \((e_x, mp) = \Delta_p\); THEN

\((e_x \otimes e_p, \Delta(mp)) = (e_x \otimes e_p, mp)\).

THE ANTIFOR 5: \( \text{Sym} \to \text{Sym} \) IS

THE AUTOMORPHISM OF \( \text{Sym} \) THAT

INTERCHANGES \( e_n \) WITH \((-1)^n h_n \), WHERE

\( h_n \) ARE THE COMPLETE S.F.'S

\[ h_n = \sum_{i=1}^{n} m_a \]

\[ 1 \leq i = n \]
A basis for \( \text{QSym} \) is given by the monomial quasi-symmetric functions

\[
M_i = \sum_{j_1 < \cdots < j_k} x_{i_1}^{j_1} x_{i_2}^{j_2} \cdots x_{i_k}^{j_k}
\]

indexed by integer compositions (ordered partitions). So, e.g.,
\[
M_{i_2} + M_{i_1} = M_{i_2}, \quad M_{i_1} = M_{i_1}
\]

As an algebra,

\[
\text{QSym} = k \left[ M_i \right] \text{ Lyndon}
\]

where \( I \) runs over "Lyndon words" in \( 1,2,3,\ldots \).

The Hopf algebra structure is given by

\[
\Delta(M_i) = \sum M_{i_1, \otimes} M_{i_2}
\]

where the sum is over strings \( I_1, I_2 \) with \( I_1 I_2 = I \) (concatenation).
This is non-cocommutative.

A formula for the antipode $S$ can be proved by induction:

$$S(M_I) = (-1)^{\ell(I)} \sum_{J \preceq I} M_J$$

where $\ell(I)$ = length of $I$

$\preceq$ is refinement order

$I$ = reverse of $I$ ($T_2 = 2$)

The dual of $Q\text{Sym}$ is a noncommutative, cocommutative Hopf algebra called $N\text{Sym}$ (noncommutative symmetric functions). They were discussed by Gel'fand et al. as an algebra,

$N\text{Sym} = \mathbb{K}\langle e_1, e_2, \ldots \rangle$

with coproduct

$$\Delta(e_i) = \sum_{j=0}^i e_j \otimes e_{i-j}$$
HOLF ALGEBRA I OF ROOTED TREES

TREES 0, 1, ▲, 1, ▲, 2, ...
(DON'T DISTINGUISH BETWEEN ▲ AND ▲)

FORESTS 0, 1, ▲, 1, ...
(DOCE IMMATERIAL)

B+: FORESTS → TREES  B+(0,1) = ▲

GRADE TREES BY NUMBER OF NON-ROOT
VERTICES: |▲| = 3

GROSSMAN-LARSON PRODUCT x ⊗ x':
LET x = B+(x₁,...,xₖ). THEN x ⊗ x'
IS THE SUM OF THE (1x₁+1) k
TREES OBTAINED BY ATTACHING THE
k BRANCHES OF x IN ALL POSSIBLE
LOCATIONS AMONG THE 1x₁+1 VERTICES
OF x'
\[ 1 \otimes 1 = \Lambda + 1 \]
\[ \Lambda \otimes 1 = \Lambda + \Lambda + \Lambda + \ldots \]
\[ 1 \otimes \Lambda = \Lambda + \Lambda + \Lambda + \ldots \]

So $\otimes$ is non-commutative.

(The one-vortex tree $\Lambda$ is the identity; the product is associative.)

If $\mathcal{I}$ is given the coproduct

\[ \Delta(\mathcal{B}^{+(x_1 \ldots x_k)}) = \sum \mathcal{B}^{+(x_i)} \otimes \mathcal{B}^{+(x_j)} \]
\[ \text{if } i \cup j = \{1, \ldots, k\} \]

then $\mathcal{I}$ is a Hopf algebra, described by Grossman and Larson.
Our basic objects are now forests; multiplication is by concatenation. The identity is the empty forest $\emptyset$.

For a tree $\mathcal{T}$, a cut is the removal of some edges. A cut is admissible if no path from root to a vertex has more than one edge cut. So for $\mathcal{T} = \_\_\_$

\[ x \times \times \times \times \]

are admissible cuts. $PC(\mathcal{T})$ is the forest parted from $\mathcal{T}$ by the cut $C$; $RC(\mathcal{T})$ is what's left.

The coproduct in $\mathcal{H}_k$ is

\[ \Delta(\mathcal{T}) = x \otimes \emptyset + \sum_{\text{admissible } \mathcal{C}} PC(\mathcal{T}) \otimes RC(\mathcal{T}) \]

and extends to forests multiplicatively.
The antipodes is given on trees by

\[ S(x) = -\sum_{\text{all cuts } c} (-1)^{\mu_c} P^c(x) R^c(x) \]

where the sum is over all cuts \( c \).

If we define an inner product on trees by

\[ (x, x') = \begin{cases} 0, & x \neq x' \\ |\text{Sym} x|, & x = x' \end{cases} \]

then there is an isomorphism \( \psi: T \rightarrow \mathbb{H}_k^* \) given by

\[ \langle \psi(x), u \rangle = (x, B + u) \]
HOPF ALGEBRA $B$ OF PLANAR ROOTED TREES

**Planar Trees** $\cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot$

**Ordered Forests** $\cdot \neq \cdot \neq \cdot, \cdot \neq \cdot \neq \cdot, \cdot$

$B^+ : \text{Ordered Forests} \rightarrow \text{Planar Trees}$ $B^+ (\cdot \cdot \cdot) = \cdot$

**Product of Planar Rooted Trees**: Let $x = B^+ (\cdot \cdot \cdot \cdot \cdot)$. Then $x \otimes x'$ is the sum of the $\binom{2|x'| - 1}{k}$ trees obtained by attaching, in order, the $k$ branches of $x$ to vertices of $x'$, respecting the natural order on the vertices of $x'$.
Although not commutative, this has the character of a shuffle product. In fact, it is an asymmetric shuffle product in the following sense: represent planar rooted trees by balanced bracket arrangements (BBR's):
All substrings of a BBA that are themselves BBA's components (correspond to branches of the planar rooted tree).

Say \( x, x' \) have corresponding BBA's \( c, c' \). Let \( c_1c_2...c_k \) be components of \( x \). Shuffle the symbols \( c_1, ..., c_k \) into the BBA \( c' \), then replace \( c_1, ..., c_k \) with corresponding BBA's for \( c = \langle c_1c_2 \rangle \), \( c' = \langle c_1c_2 \rangle \).

For \( c = \langle c_1c_2 \rangle \), \( c' = \langle c_1c_2 \rangle \) we have

\[
\begin{align*}
   c \cdot c' &= c_1c_2\langle c_1c_2 \rangle + c_1c_2\langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle \\
   &= \langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle \\
   &= \langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle + \langle c_1c_2 \rangle
\end{align*}
\]

(Asymmetric since only components of the left-hand factor are kept together while shuffling.)
FOISSY HOPF ALGEBRA $H_F$

Our basic objects are ordered forests of planar rooted trees; juxtaposition product is now non-commutative, coproduct is that for Kreimer’s Hopf algebra

$$\Delta(x) = x \otimes x + \sum_{\text{admissible}} p^{c(x)} \otimes R^{c(x)}$$

Note that $p^{c(x)}$ can be ordered by using the order of the “new” root vertices; all planar rooted trees have a natural numbering of vertices:

![Diagram]

Antipode is

$$S(x) = - \sum (-1)^{|c|} p^{c(x)} \otimes R^{c(x)}$$
By some proof as before, \( P \cong H^*_p \)
but in fact \( H^*_p \cong H_1 \); Poisson gave an inner product on \( H^*_p \)
with \( (F_G, H) = (F \otimes G, \Delta(H)) \).
If we define \( e_F \) by
\[
(e_F, G) = \delta_{FG}
\]
then the function \( x \mapsto e_{B_-(x)} \)
(where \( B_- = B_+^{-1} \)) is an isomorphism. For example,
\[
e_{e_{e_o}} = e_{e_o} = e_1 + 2e_{e_o}
\]
since \( e_{e_o} = e_o \), \( e_{e_o} = \mathbb{1} \) and \( e_1 = \mathbb{1} - 2\mathbb{1} \);
this corresponds to
\[
1 \otimes 1 = 2\mathbb{1} + \mathbb{1}.
\]
A COMMUTATIVE DIAGRAM

let \( l_k \) be the "ladder" \( B_+^{k-1}(\ast) \)

THE KRENER COPRODUCT FORMULA SHOWS THE \( l_k \) ARE DIVIDED POWERS, SO THERE IS A HOPF ALGEBRA MAP

\[
\text{Sym} \xrightarrow{l} H_k
\]

SENDING \( e_k \) TO \( l_k \). IN FACT, THE DIAGRAM

\[
\begin{array}{ccc}
\text{NSym} & \xrightarrow{l} & H_f \\
\pi \downarrow & & \downarrow \phi \\
\text{Sym} & \xrightarrow{l} & H_k
\end{array}
\]

COMMUTES, WHERE \( \pi : \text{NSym} \to \text{Sym} \)

ABELIANIZE AND \( \phi : H_f \to H_k \)

FORGETS ORDER.
THIS IS DUALIZED TO

\[ Q_{\text{sym}} \leftarrow \mathcal{L} \rightarrow \mathcal{H}_k \]

\[ U \uparrow \phi^* \]

\[ \text{Sym} \leftarrow \mathcal{L} \rightarrow \mathcal{T} \]

Here, \( \mathcal{L} \) sends \( B_+(l_{i_1} \ldots l_{i_n}) \) to \( \mathcal{T} \) to the monomial quasisymmetric function \( m_{i_1 \ldots i_k} \), and all other trees to 0; \( \phi^* \) sends a rooted tree to a sum of planar rooted trees given by "exercising" \( \times \) by permuting branches out of all its vertices:

\[ \phi^*(\Uparrow) = 2 \Uparrow + 2 \Uparrow + 2 \Uparrow, \]

and \( \mathcal{L} \) sends \( B_+(l_{i_1} \ldots l_{i_n}) \) to \( \mathcal{T} \) to

\[ \sum_{j_{i_1}, j_{i_2} \ldots j_{i_n}} \hat{m}_{j_{i_1} j_{i_2} \ldots j_{i_n}}, \] and all other trees to 0.
\[ 2M_{211} + 2M_{121} + 2M_{112} \rightarrow 2e^+ + 2\bar{\nu} + 2\nu^+ \]

\[ 2m_{211} \leftarrow e^+ \uparrow \]