Multiple Zeta Values, Iterated Integrals, and Labeled Posets

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Outline

Introduction

Multiple zeta values (MZVs) are defined by

$$\zeta(i_1, \ldots, i_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}$$

for positive integers $i_1, \ldots, i_k$ with $i_1 > 1$. The sum of the indices $i_1 + \cdots + i_k$ is called the weight, and k is the depth. For $k \leq 2$ the study of these numbers goes back to Euler, but the general depth case emerged as an area of active research only in the 1990's. At that time they appeared simultaneously in knot theory and perturbative quantum field theory. They have appeared in more areas since, and have been the subject of hundreds of papers.

Introduction cont'd

I had the good fortune to be one of the first people to write about MZVs, and have now been thinking about them for over thirty years. Nevertheless, sometimes a new idea appears that makes me feel like I've had blinders on for decades. I will talk about one such idea today: a way of representing iterated integrals (and thus MZVs) graphically as labeled posets. I learned of this from Shuji Yamamoto and give his formulation here, although it seems that Jean Ecalle developed similar ideas earlier.

MZVs as sums and integrals

First I will describe how I thought of MZVs before seeing Yamamoto's idea. MZVs can be represented by iterated integrals as well as by series. For example,

$$\int_0^1 \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_1}{1-t_1} = \int_0^1 \frac{dt_3}{t_3} \int_0^{t_3} \sum_{i\ge 1} \frac{t_2^i}{i} \frac{dt_2}{1-t_2} = \int_0^1 \sum_{i,j\ge 1} \frac{t_3^{i+j}}{i(i+j)} \frac{dt_3}{t_3} = \sum_{i,j\ge 1} \frac{1}{i(i+j)^2} = \zeta(2,1).$$

Algebraic notation

We can represent iterated integral computations like the preceding by using the notation

$$x \sim \frac{dt}{t}, \quad y \sim \frac{dt}{1-t}$$

so that the form integrated on the preceding slide is xy^2. We can then think of ζ as the function that evaluates the iterated integral, so $\zeta(2,1) = \zeta(xy^2)$, and more generally

$$\zeta(i_1, \dots, i_k) = \zeta(x^{i_1 - 1} y \cdots x^{i_k - 1} y).$$

The monomials "live" in the noncommutative polynomial ring $\mathbb{Q}\langle x, y\rangle$. The change of variable $t \mapsto 1-t$ corresponds to the antiautomorphism τ of $\mathbb{Q}\langle x, y\rangle$ that exchanges x and y, so e.g., $\zeta(3,2) = \zeta(x^2yxy) = \zeta(\tau(x^2yxy)) = \zeta(xyxy^2) = \zeta(2,2,1)$.

Algebraic notation cont'd

This is the "duality" of MZVs. Actually the iterated integral only converges if the monomial begins in x and ends in y. Let $\mathfrak{H}$ be the underlying rational vector space of $\mathbb{Q}\langle x, y\rangle$, $\mathfrak{H}^0$ the subspace generated by monomials starting with x and ending in y, together with the empty monomial 1. There is a commutative product on $\mathfrak{H}$, namely the shuffle product $\sqcup\!\sqcup$, and $(\mathfrak{H}^0, \sqcup\!\sqcup)$ is a subalgebra of $(\mathfrak{H}, \sqcup\!\sqcup)$. We have, e.g.,

$$xy \sqcup\!\sqcup\, xy = 2xyxy + 4x^2y^2. \tag{1}$$

In fact shuffle product corresponds, via integration by parts, to the product of iterated integrals, so ζ becomes a homomorphism from $(\mathfrak{H}^0, \sqcup\!\sqcup)$ to the reals.

The other product

Thus corresponding to Eq. (1) we have

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1) \qquad (2)$$

But we can also multiply MZVs as series, e.g.,

$$\zeta(2)^2 = \zeta(4) + 2\zeta(2,2). \qquad (3)$$

(This is sometimes called the "stuffle" product.) Since $\zeta(2)^2 = \frac{\pi^4}{36} = \frac{5}{2}\zeta(4)$, this implies that $\zeta(2,2) = \frac{3}{4}\zeta(4)$. Comparing Eq. (3) with Eq. (2) gives $\zeta(4) = 4\zeta(3,1)$, or $\zeta(3,1) = \frac{1}{4}\zeta(4)$. More generally, playing the shuffle product against the stuffle product gives relations of MZVs (conjecturally all relations).

Multiple zeta-star values

In addition to multiple zeta values, there are multiple zeta-star values (MZSVs):

$$\zeta^\star(i_1, \dots, i_k) = \sum_{n_1 \ge \cdots \ge n_k \ge 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}.$$

These differ from MZVs only in having non-strict inequalities in the summation indices. Thus, e.g.,

$$\zeta^\star(3,1,2) = \zeta(3,1,2) + \zeta(4,2) + \zeta(3,3) + \zeta(6). \qquad (4)$$

Here is a recipe for going from MZVs to MZSVs, based on our algebraic notation: take the monomial in x and y, replace every y except the last by $x + y$, and expand.

Multiple zeta-star values cont'd

Thus, since $\zeta(3,1,2) = \zeta(x^2y^2xy)$,

$$\begin{aligned}\zeta^\star(3,1,2) &= \zeta(x^2(x+y)^2xy) = \zeta(x^2(x^2+xy+yx+y^2)xy)\\ &= \zeta(x^5y)+\zeta(x^3yxy)+\zeta(x^2yx^2y)+\zeta(x^2y^2xy)\\ &= \zeta(6)+\zeta(4,2)+\zeta(3,3)+\zeta(3,1,2),\end{aligned}$$

which agrees with Eq. (4) on the preceding slide. Note that this expansion (which we shall call the collapsing sum) of an MZSV of depth d has 2^{d-1} terms.

The sum theorem

Besides the duality mentioned above, MZVs also satisfy the sum theorem. This says that the sum of all MZVs of weight n and a fixed depth $d < n$ is $\zeta(n)$. For example,

$$\zeta(4,1,1)+\zeta(3,2,1)+\zeta(3,1,2)+\zeta(2,3,1)+\zeta(2,2,2)+\zeta(2,1,3) = \zeta(6).$$

The duality theorem fails for MZSVs: there is no simple relation between $\zeta^{\star}(w)$ and $\zeta^{\star}(\tau(w))$. But the sum theorem holds in the following form: the sum of all MZSVs of weight n and depth d is $\binom{n-1}{d-1}\zeta(n)$.

The MZV landscape

Before going further, here are some general remarks on the MZV landscape. All known relations of MZVs preserve weight, so we think of the algebra of MZVs as graded. It is conjectured that the dimension of the rational vector space $\mathcal{MZV}_n$ of weight-n MZVs is P_n, the nth Padovan number, defined by $P_1 = 0$, $P_2 = P_3 = 1$, and $P_n = P_{n-2} + P_{n-3}$ for $n \geq 4$. While putting a lower bound on $\dim_{\mathbb{Q}} \mathcal{MZV}_n$ seems out of reach, it is known that $\dim_{\mathbb{Q}} \mathcal{MZV}_n \leq P_n$.

Back in 1997 I conjectured that $\mathcal{MZV}_n$ has basis H_n, where H_n is the set of weight-n MZVs with only arguments 2 or 3, e.g., $H_7 = \{\zeta(3,2,2), \zeta(2,3,2), \zeta(2,2,3)\}$; H_n is easily seen to have cardinality P_n. In 2011 Francis Brown proved that H_n spans $\mathcal{MZV}_n$. It is an open question whether the corresponding set of MZSV's spans.

The MZV landscape cont'd

We call an MZV reducible if it is a rational polynomial in the ordinary zeta values $\zeta(i)$. All MZVs of weight ≤ 7 are reducible, but in weight 8, $\zeta(6,2)$ appears not to be. In any case one can't form $P_8 = 4$ independent quantities by taking products of zeta values of lesser weight; all weight-8 products of zeta values are rational linear combinations of $\{\zeta(8), \zeta(2)\zeta(3)^2, \zeta(3)\zeta(5)\}$. All MZVs of weight 9 are reducible, but in weights 10 and higher this isn't the case. All MZVs of the form $\zeta(n, \underbrace{1, \dots, 1}_{k})$ ("height one") are reducible, as are all double zeta values $\zeta(a, b)$ with $a + b$ odd. (The latter result goes back, in essence, to Euler.)

Yamamoto's formalism

Now I will describe Yamamoto's idea. He considers iterated integrals over some subset of $[0,1]^n$, involving the two forms

$$\omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1-t}$$

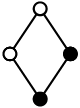
(what we called x and y above). Let (X, δ) be a 2-labeled poset, i.e., a finite partially ordered set X with a function $\delta : X \to \{0, 1\}$. Call (X, δ) admissible if $\delta(x) = 1$ for all minimal $x \in X$ and $\delta(x) = 0$ for all maximal $x \in X$. For an admissible 2-labeled poset (X, δ), define the associated integral by

$$I(X) = \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta(x)}(t_x), \tag{5}$$

where $\Delta(X) = \{(t_x)_{x\in X} \in [0,1]^X \mid t_x < t_y \text{ if } x < y \text{ in } X\}$.

Graphical representation

We can represent a 2-labeled poset (X, δ) graphically by its Hasse diagram, with an open dot $\circ$ for those elements $x \in X$ with $\delta(x) = 0$, and a closed dot $\bullet$ for those $x \in X$ with $\delta(x) = 1$. For example, the 2-labeled poset $X = \{x_1, x_2, x_3, x_4\}$ with $x_1 > x_2 > x_4$, $x_1 > x_3 > x_4$, $\delta(x_1) = \delta(x_2) = 0$, and $\delta(x_3) = \delta(x_4) = 1$ has graphical representation



and associated integral

$$I(X) = \int_{\substack{t_1 > t_2 > t_4 \\ t_1 > t_3 > t_4}} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{dt_3}{1 - t_3} \frac{dt_4}{1 - t_4}.$$

2-labeled posets and MZVs

Admissibility of (X, δ) guarantees convergence of the integral $I(X)$. If X is a chain, then $I(X)$ is just the iterated integral representation of a multiple zeta value as described above. For example,

$I\Bigg($  $\Bigg)$

$$= \int_{t_1<t_2<t_3<t_4<t_5<t_6} \frac{dt_6}{t_6}\frac{dt_5}{t_5}\frac{dt_4}{1-t_4}\frac{dt_3}{1-t_3}\frac{dt_2}{t_2}\frac{dt_1}{1-t_1} =$$

$$\int_0^1 \frac{dt_6}{t_6}\int_0^{t_6}\frac{dt_5}{t_5}\int_0^{t_5}\frac{dt_4}{1-t_4}\int_0^{t_4}\frac{dt_3}{1-t_3}\int_0^{t_3}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}$$

$$= \zeta(x^2y^2xy) = \zeta(3,1,2).$$

Properties of the integral

Yamamoto proved the following theorem, which follows easily from general properties of iterated integrals.

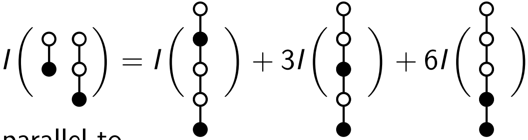
Theorem

Let X be an admissible 2-labeled poset.

1. *If Y is another admissible 2-labeled poset, $I(X \amalg Y) = I(X)I(Y)$.*
2. *If $a, b \in X$ are incomparable, let $X_{a<b}$ be X with the additional relation $a < b$. Then $I(X) = I(X_{a<b}) + I(X_{b<a})$.*
3. *If $X^\vee$ is X with reversed order and new labeling function $\delta^\vee(x) = 1 - \delta(x)$, then $I(X^\vee) = I(X)$.*

Property 2 and shuffle product

It is evident that combining two disjoint chains via Property 2 is equivalent to shuffle product in $\mathfrak{H}$. For example,



is exactly parallel to

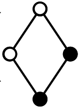
$$xy \sqcup x^2y = xyx^2y + 3x^2yxy + 6x^3y^2,$$

both showing that

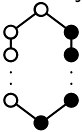
$$\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1).$$

Using Property 2

In fact, use of Property 2 allows us to write $I(X)$ as a sum of multiple zeta values for any admissible 2-labeled poset X. For example,

$I\Big($  $\Big) = I\Big($  $\Big) + I\Big($  $\Big) = \zeta(3,1) + \zeta(2,2).$

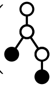
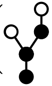
By the way, the sum theorem for MZVs implies that the latter sum is $\zeta(4)$, and more generally that

$I\Big($  $\Big) = \zeta(n),$

where the diagram has $n-k$ open dots and k closed ones, for $2 \le k \le n-2$.

Property 3

Property 3 follows from the change of variable $t \mapsto 1-t$ in the iterated integral and generalizes the duality of MZVs. It says the value of I on a labeled poset is unchanged if the poset is "flipped over" and the labels reversed, e.g.,

$I\Bigg($  $\Bigg) = I\Bigg($  $\Bigg).$

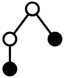
We note that the weight of a given diagram is simply the number of dots, and the depth is the number of closed dots. All Property 2 transformations preserve weight and depth, while Property 3 takes a diagram of weight n and depth d to a diagram of weight n and depth $n-d$.

Mordell-Tornheim sums

An advantage of Yamamoto's formalism is that it makes it easy to see how various zeta functions can be written in terms of MZVs. For example, the Mordell-Tornheim sums

$$T(n_1, n_2, \dots, n_k; p) = \sum_{m_1,\dots,m_k \ge 1} \frac{1}{m_1^{n_1} \cdots m_k^{n_k} (m_1 + \cdots + m_k)^p}$$

can be expressed as integrals associated with 2-labeled posets, as shown by the example

$I\Big($  $\Big)$

$$= \int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_2}{1-t_2} \int_0^{t_4} \frac{dt_1}{1-t_1} =$$

$$\int_0^1 \frac{dt_4}{t_4} \sum_{i,j\ge 1} \frac{t_4^{i+j}}{i^2 j} = \sum_{i,j\ge 1} \frac{1}{i^2 j(i+j)} = T(2,1;1).$$

Mordell-Torheim sums cont'd

If we compare the shuffle product with the poset representing Mordell-Tornheim sums, it is easy to see that

$$T(n_1, n_2, \ldots, n_k; p) = \zeta(x^p(x^{n_1-1}y \sqcup\!\sqcup x^{n_2-1}y \sqcup\!\sqcup \cdots \sqcup\!\sqcup x^{n_k-1}y)), \quad (6)$$

which gives a succinct statement of how Mordell-Tornheim sums can be expanded into MZVs. For example,

$$T(2,1;3) = \zeta(x^3(xy \sqcup\!\sqcup y)) = \zeta(x^3(yxy + 2xy^2)) = \zeta(x^3yxy + 2x^4y^2) = \zeta(4,2) + 2\zeta(5,1).$$

Before seeing Yamamoto's idea, I'd been expanding Mordell-Torhneim sums into MZVs for years without recognizing Eq. (6).

Corollaries

Two corollaries of Eq. (6) are the following. First, if $k = 2$ we can expand out the shuffle product to get the following explicit (and well-known) formula:

$$T(n_1, n_2; p) = \sum_{i=0}^{n_2-1} \binom{n_1+i-1}{i} \zeta(p+n_1+i, n_2-i) + \sum_{j=0}^{n_1-1} \binom{n_2+j-1}{j} \zeta(p+n_2+j, n_1-j). \quad (7)$$

Second, if $n_1 = n_2 = \cdots = n_k = 1$ we have

$$T(\underbrace{1,\dots,1}_{k}; p) = k!\zeta(x^p y^k) = k!\zeta(p+1, \underbrace{1,\dots,1}_{k-1}).$$

Corollaries cont'd

Using Eq. (6), one can give a formula for $T(n_1, n_2, n_3; p)$ in terms of MZVs by expanding out the shuffle product

$$x^p(x^{n_1-1}y ⧢ x^{n_2-1}y ⧢ x^{n_3-1}y),$$

but it's a daunting array of binomial and trinomial coefficients that won't fit on a slide; it has

$$\binom{n_1+n_2+n_3}{n_1, n_2, n_3}$$

terms in all. This results in a formula drastically more complicated than Eq. (7).

Multiple zeta-star values

The multiple zeta-star values are easily written in Yamamoto's formalism; in fact

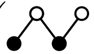
$$\zeta^\star(k_1,\dots,k_r) = I(X),$$

where X is the poset

$$\{\bar{x}_1 < x_2 < \cdots < x_{k_1} > \bar{x}_{k_1+1} < x_{k_1+2} < \cdots < x_{k_1+k_2} > \cdots > \bar{x}_{k_1+\cdots+k_{r-1}+1} < x_{k_1+\cdots+k_{r-1}+2} < \cdots < x_{k_1+\cdots+k_r}\},$$

with barred elements having label 1 and all others having label 0. For example,

$$I\left(\quad \right) = \int_{t_1<t_2>t_3<t_4} \frac{dt_4}{t_4}\frac{dt_3}{1-t_3}\frac{dt_2}{t_2}\frac{dt_1}{1-t_1}$$



Multiple zeta-star values cont'd

$$= \int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1-t_3} \int_{t_3}^1 \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_1}{1-t_1}$$

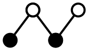
$$= \int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1-t_3} \int_{t_3}^1 \sum_{i\ge 1} \frac{t_2^{i-1}}{i} dt_2 =$$

$$\int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1-t_3} \sum_{i\ge 1} \frac{1-t_3^i}{i^2} = \int_0^1 \sum_{i,j\ge 1} \left[\frac{t_4^{j-1}}{i^2 j} - \frac{t_4^{i+j-1}}{i^2(i+j)} \right] dt_4$$

$$= \sum_{i=1}^\infty \frac{1}{i^2} \sum_{j=1}^\infty \left[\frac{1}{j^2} - \frac{1}{(i+j)^2} \right] = \sum_{i=1}^\infty \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j^2} = \zeta^\star(2,2).$$

Property 2 again

Applying Property 2 of Yamamoto's theorem to the representation of $\zeta^\star(2,2)$ gives

$I\Big($  $\Big) = I\Big($  $\Big) + 4I\Big($  $\Big),$

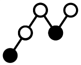
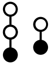
implying $\zeta^\star(2,2) = \zeta(2,2) + 4\zeta(3,1)$; this differs from the collapsing sum $\zeta^\star(2,2) = \zeta(2,2) + \zeta(4)$. As we noted earlier, Property 2 preserves depth. Comparing the two expansions in this case gives $\zeta(3,1) = \frac{1}{4}\zeta(4)$, which we obtained earlier by comparing stuffle and shuffle products.

Cut and subtract

Now we undertake some detailed computations of multiple zeta-star values using Yamamoto's representation. A key move, which we call "cut and subtract", uses Property 2 in the form

$$I(X_{a<b}) = I(X) - I(X_{b<a}),$$

where X is a disjoint union to which Property 1 can be applied. For example,

$\zeta^\star(3,2) = I\Big($  $\Big) = I\Big($  $\Big) - I\Big($  $\Big)$

$= I\Big($  $\Big) I\Big($  $\Big) - I\Big($  $\Big) = \zeta(3)\zeta(2) - \zeta(2,3).$

While this particular result is rather trivial, we will make frequent use of cut and subtract in the slides that follow.

MZSVs of large depth

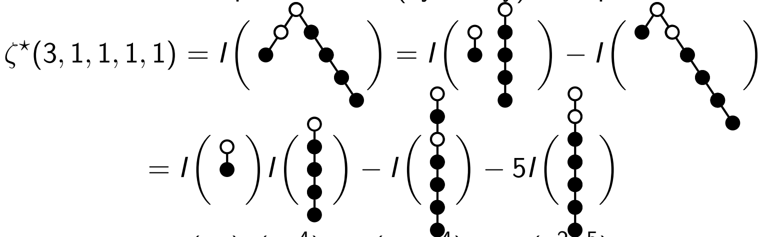
We recall that the collapsing sum of an MZSV of depth d has 2^{d-1} terms. Thus, calculating an MZSV whose depth is close to its weight seems onerous. The highest depth is one less than the weight, and here the sum theorem comes to the rescue in evaluating the single MZSV, e.g.,

$$\zeta^\star(2,1,1,1,1,1) = \binom{6}{5}\zeta(7) = 6\zeta(7)$$

But what about an MZSV of next-to-greatest depth, e.g., $\zeta(3,1,1,1,1)$? Its collapsing sum has 16 terms.

MZSVs of large depth cont'd

But if we instead use Yamamoto's representation, together with cut and subtract, we can write it as a product minus a sum of MZVs of depth 5, hence (by dualiy) of depth 2:

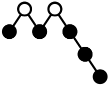
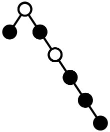


$$
\begin{aligned}
&= \zeta(xy)\zeta(xy^4) - \zeta(xyxy^4) - 5\zeta(x^2y^5) \\
&= \zeta(xy)\zeta(x^4y) - \zeta(x^4yxy) - 5\zeta(x^5y^2) \\
&= \zeta(2)\zeta(5) - \zeta(5,2) - 5\zeta(6,1) \\
&= \zeta(7) + \zeta(2,5) - 5\zeta(6,1),
\end{aligned}
$$

which is far shorter than the 16-term collapsing sum!

MZSVs of large depth cont'd

Here is another example:

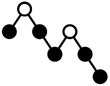
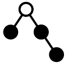
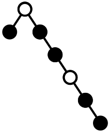
$\zeta^\star(2,2,1,1,1) = I\Big($  $\Big) = I\Big($   $\Big) - I\Big($  $\Big)$

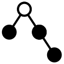
$= I\Big($  $\Big) I\Big($  $\Big) - 2I\Big($  $\Big) - 4I\Big($  $\Big)$

$$=2\zeta(xy^2)\zeta(xy^3) - 2\zeta(xy^2xy^3) - 4\zeta(xyxy^4)$$
$$=2\zeta(x^2y)\zeta(x^3y) - 2\zeta(x^3yx^2y) - 4\zeta(x^4yxy)$$
$$=2\zeta(3)\zeta(4) - 2\zeta(4,3) - 4\zeta(5,2)$$
$$=2\zeta(7) + 2\zeta(3,4) - 4\zeta(5,2).$$

MZSVs of large depth cont'd

And yet another:

$\zeta^\star(2,1,2,1,1) = I\Big($  $\Big) = I\Big($   $\Big) - I\Big($  $\Big)$

$= I\Big($  $\Big) I\Big($  $\Big) - 3I\Big($  $\Big) - 3I\Big($  $\Big)$

$$=3\zeta(xy^3)\zeta(xy^2) - 3\zeta(xy^3xy^2) - 3\zeta(xy^2xy^3)$$

$$=3\zeta(x^3y)\zeta(x^2y) - 3\zeta(x^2yx^3y) - 3\zeta(x^3yx^2y)$$

$$=3\zeta(4)\zeta(3) - 3\zeta(3,4) - 3\zeta(4,3)$$

$$=3\zeta(7).$$

MZSVs of large depth cont'd

Calculations like these give simple formulas for all $n-2$ MZSVs of weight n and depth $n-2$. The general result is

$$\zeta^\star(3,\{1\}_{n-3}) = \zeta(n) + \zeta(2,n-2) - (n-2)\zeta(n-1,1)$$

(where $\{1\}_p$ means p repetitions of 1) and

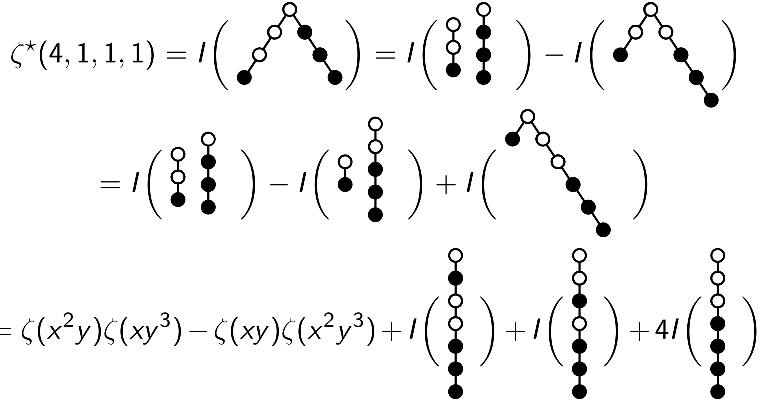
$$\zeta^\star(2,\{1\}_{r-1},2,\{1\}_{n-3-r}) = (r+1)\zeta(n) + (r+1)\zeta(r+2,n-2-r) - (n-2-r)\zeta(n-1-r,r+1) \quad (8)$$

for $1 \le r \le n-3$. If $n \ge 5$ is odd there is exactly one case where the last two terms in Eq. (8) cancel, giving

$$\zeta^\star(2,\{1\}_{\frac{n-5}{2}},2,\{1\}_{\frac{n-3}{2}}) = \frac{n-1}{2}\zeta(n).$$

MZSVs of large depth cont'd

This method continues to be useful at the next lower depth, using cut and subtract twice. For example,



MZSVs of large depth cont'd

$$
\begin{aligned}
=&\zeta(x^2y)\zeta(xy^3) - \zeta(xy)\zeta(x^2y^3) + \zeta(xyx^2y^3) + \zeta(x^2yxy^3)\\
&+ 4\zeta(x^3y^4)\\
=&\zeta(x^2y)\zeta(x^3y) - \zeta(xy)\zeta(x^3y^2) + \zeta(x^3y^2xy) + \zeta(x^3yxy^2)\\
&+ 4\zeta(x^4y^3)\\
=&\zeta(3)\zeta(4) - \zeta(2)\zeta(4,1) + \zeta(4,1,2) + \zeta(4,2,1) + 4\zeta(5,1,1)\\
=&\zeta(7) + \zeta(3,4) - \zeta(6,1) - \zeta(2,4,1) + 4\zeta(5,1,1).
\end{aligned}
$$

This 5-term formula is better than the 8-term collapsing sum for $\zeta^\star(4,1,1,1)$. It is also better than the 10-term formula

$$
\zeta^\star(4,1,1,1) = \sum_{a+b+c=6,\ a,b,c\geq 1} a\zeta(a+1,b,c) \qquad (9)
$$

obtained by using Property 2 alone.

Scaling by weight

The difference becomes more dramatic at higher weights (We've been keeping the weight at 7 so the diagrams fit on a slide). The 5-term formula just obtained generalizes to

$$\zeta^\star(4,\{1\}_{n-4}) = \zeta(n)+\zeta(3,n-3)-\zeta(n-1,1)-\zeta(2,n-3,1) + (n-3)\zeta(n-2,1,1),$$

in weight n, where $\{1\}_k$ means k repetitions of 1. On the other hand, Eq. (9) generalizes to

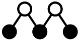
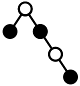
$$\zeta^\star(4,\{1\}_{n-4}) = \sum_{a+b+c=n-1,\ a,b,c\geq 1} a\zeta(a+1,b,c)$$

with $\binom{n-2}{2}$ terms, and the collapsing sum for $\zeta^\star(4,\{1\}_{n-4})$ has 2^{n-4} terms. In weight 11, these numbers are 5, 36, and 128.

Another set of identities

The computations

$\zeta^\star(2,1) = I\Big($  $\Big) = 2I\Big($  $\Big) = 2\zeta(xy^2) = 2\zeta(x^2y) = 2\zeta(3),$

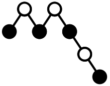
$\zeta^\star(2,2,1) = I\Big($  $\Big) = I\Big($   $\Big) - I\Big($  $\Big)$

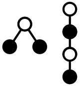
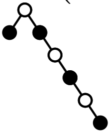
$= I\Big($  $\Big) I\Big($  $\Big) - 2I\Big($  $\Big) - 2I\Big($  $\Big)$

$$\begin{aligned} &= \zeta^\star(2,1)\zeta(2) - 2\zeta(2,3) - 2\zeta(3,2) \\ &= 2\zeta(3)\zeta(2) - 2\zeta(2,3) - 2\zeta(3,2) \\ &= 2\zeta(5), \end{aligned}$$

Another set of identities cont'd

and

$\zeta^\star(2,2,2,1) = I\Big($  $\Big) = I\Big($  $\Big) - I\Big($  $\Big)$

$= I\Big($  $\Big) I\Big($  $\Big) - I\Big($  $\Big) + I\Big($  $\Big) =$

$\zeta^\star(2,2,1)\zeta(2) - \zeta^\star(2,1)\zeta(2,2) + 2I\Big($  $\Big) + 2I\Big($  $\Big) + 2I\Big($  $\Big)$

$$= 2\zeta(5)\zeta(2) - 2\zeta(3)\zeta(2,2) + 2\zeta(2,2,3) + 2\zeta(2,3,2) + 2\zeta(3,2,2)$$

Another set of identities cont'd

$$
\begin{aligned}
&=2\zeta(7)+2\zeta(5,2)+2\zeta(2,5)-2\zeta(5,2)-2\zeta(2,5)-2\zeta(3,2,2)\\
&-2\zeta(2,3,2)-2\zeta(2,2,3)+2\zeta(2,2,3)+2\zeta(2,3,2)+2\zeta(3,2,2)\\
&=2\zeta(7)
\end{aligned}
$$

show how the family of identities

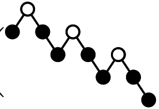
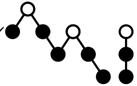
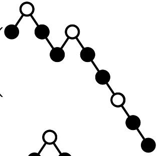
$$\zeta^\star(\{2\}_n,1)=2\zeta(2n+1)$$

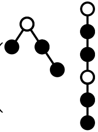
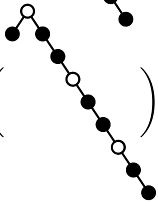
can be proved by induction.

The pattern continues . . .

But this is just part of a larger pattern. Recalling our earlier computation $\zeta^\star(2,1,2,1,1) = 3\zeta(7)$, we have

$$\zeta^\star(2,1,2,1,2,1,1) =$$

$I\Bigg($  $\Bigg) = I\Bigg($  $\Bigg) - I\Bigg($  $\Bigg)$

$= \zeta^\star(2,1,2,1,1)\zeta(3) - I\Bigg($  $\Bigg) + I\Bigg($  $\Bigg)$

The pattern continues . . .

$$= 3\zeta(7)\zeta(3) - 3\zeta(4)\zeta(3,3) + 3I\Big($$  $$\Big) + 3I\Big($$  $$\Big) + 3I\Big($$  $$\Big)$$

$$=3\zeta(7)\zeta(3) - 3\zeta(4)\zeta(3,3) + 3\zeta(3,3,4) + 3\zeta(3,4,3) + 3\zeta(4,3,3)$$
$$=3\zeta(7)\zeta(3) - 3\zeta(7,3) - 3\zeta(3,7) = 3\zeta(10),$$

which shows how the identity

$$\zeta^{\star}(\{2,1\}_n, 1) = 3\zeta(3n+1)$$

can be establised by induction. This generalizes to

$$\zeta^{\star}(\{2,\{1\}_p\}_n, 1) = (p+2)\zeta((p+2)n+1). \qquad (10)$$

Another proof

The general identity (10) also follows from the cyclic sum theorem for multiple zeta-star values of Y. Ohno and N. Wakabayashi (*Acta Arithmetica* 2006). In general the cyclic sum theorem gives formulas for certain sums of MZSVs as multiples of zeta values rather than expressions for single MZSVs. But our method and the cyclic sum theorem coincide with identities like (10) that give a single MZSV as a multiple of a single zeta value.

An algebra of 2-labeled posets

We now consider some extensions of Yamamoto's idea. Our first extension arises naturally from putting Yamamoto's idea into an algebraic context. Define a graded $\mathbb{Q}$-algebra $\mathcal{A}$ as follows. For each 2-labeled poset X, $\mathcal{A}$ has a generator $[X]$ in degree $\operatorname{card} X$. We agree that $[X] = [Y]$ if X and Y are isomorphic as 2-labeled posets, and define the product by $[X][Y] = [X \amalg Y]$. If $\mathcal{A}^0 \subset \mathcal{A}$ is the subalgebra generated by admissible 2-labeled posets, then Yamamoto's theorem implies that $I : \mathcal{A}^0 \to \mathbb{R}$ is a homomorphism.

Call a 2-labeled poset X lower admissible if $\delta(x) = 1$ for every minimal element of X. Then the subspace $\mathcal{A}^1$ of $\mathcal{A}$ generated by lower admissible 2-labeled posets is a subalgebra, and in fact $\mathcal{A}^0 \subset \mathcal{A}^1 \subset \mathcal{A}$.

A homomorphism from $\mathcal{A}$ to $\mathfrak{H}$

Now define a $\mathbb{Q}$-linear map $J : \mathcal{A} \to \mathfrak{H}$ by sending any chain $\{x_1 > x_2 > \cdots > x_n\}$ to the monomial $a(x_1)a(x_2)\cdots a(x_n)$, where $a(x_i) = x$ if $\delta(x_i) = 0$ and $a(x_i) = y$ if $\delta(x_i) = 1$. In general define $J(P)$ as $\sum_i J(c_i)$, where $\sum_i c_i$ is the formal sum of all chains obtained by adding more relations to P.

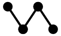
Let $\mathfrak{H}^1$ be the subspace of $\mathfrak{H}$ generated by 1 and monomials ending in y. Note that $(\mathfrak{H}^1, ⧢)$ is a commutative algebra. Then J restricted to $\mathcal{A}^1$ is a homomorphism $J : \mathcal{A}^1 \to \mathfrak{H}^1$. Similarly, if $\mathfrak{H}^0$ is the subspace of $\mathfrak{H}^1$ generated by 1 and all monomials that start in x and end in y, then $(\mathfrak{H}^0, ⧢)$ is an algebra and $J : \mathcal{A}^0 \to \mathfrak{H}^0$ is a homomorphism such that $I(X) = \zeta(J(X))$.

Zigzags

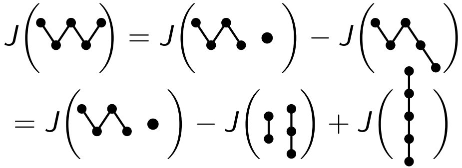
The homomorphism $J : \mathcal{A}^1 \to \mathfrak{H}^1$ has some interesting combinatorics, as can be seen from the following family of examples:

$J(\bullet) = y$, $J\Big($  $\Big) = y^2$,

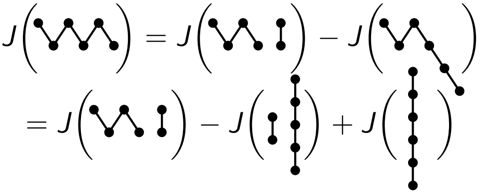
$J\Big($  $\Big) = J\Big($  $\Big) - J\Big($  $\Big) = y^2 ⧢ y - y^3 = 2y^3$,

$J\Big($  $\Big) = J\Big($  $\Big) - J\Big($  $\Big) = y^2 ⧢ y^2 - y^4 = 5y^4$

Zigzags cont'd



$$= 5y^4 ⧢ y - y^2 ⧢ y^3 + y^5 = 16y^5,$$



$$= 5y^4 ⧢ y^2 - y^2 ⧢ y^4 + y^6 = 61y^6.$$

Zigzags cont'd

The sequence $1, 1, 2, 5, 16, 61, \dots$ is that of Euler or zigzag numbers, i.e., the number of permutations of $\{1, 2, \dots, n\}$ that follow a down/up pattern (for $n \geq 2$, this is half the number of alternating permutations of $\{1, 2, \dots, n\}$). If we let e_n be the nth such number, then J applied to an n-vertex zigzag pattern as above is evidently $e_n y^n$. The calculations above give the recurrence

$$e_n = \begin{cases} \sum_{i=1}^{\frac{n}{2}} (-1)^{i-1} e_{n-2i} \binom{n}{2i}, & n \text{ even}, \\ \sum_{i=1}^{\frac{n+1}{2}} (-1)^{i-1} e_{n+1-2i} \binom{n}{2i-1}, & n \text{ odd}, \end{cases}$$

where we set $e_0 = 1$.

Cone construction

Given 2-labeled posets $X_1, X_2, \ldots, X_n$, form a new poset

$$B_+(X_1 X_2 \cdots X_n) = X_1 \amalg X_2 \amalg \cdots \amalg X_n \amalg \{p\}$$

with $p > x$ for $x \in X_i$, $1 \le i \le n$. The labeling on $B_+(X_1 \cdots X_n)$ is the disjoint union of those on the X_i, with p getting label 0. Then $B_+(X_1 \cdots X_n)$ is admissible if each X_i is lower-admissible, and its image under J is as follows.

Proposition

For $X_1, X_2, \ldots, X_n \in \mathcal{A}^1$,

$$J(B_+(X_1 \cdots X_n)) = x(J(X_1) ш \cdots ш J(X_n)) \in \mathfrak{H}^0.$$

Cone construction cont'd

Then we can think of Eq. (6) above as

$$T(n_1,\ldots,n_k;p)=\zeta(J(B_+^p(B_+^{n_1-1}(\bullet)\cdots B_+^{n_k-1}(\bullet)))).$$

By the way, other types of Mordell-Tornheim sums can be expressed similarly. For example,

$$T(n_1,n_2,n_3;p;q)=$$
$$\sum_{m_1,m_2,m_3\geq 1}\frac{1}{m_1^{n_1}m_2^{n_2}m_3^{n_3}(m_1+m_2)^p(m_1+m_2+m_3)^q}$$
$$=\zeta(J(B_+^q(B_+^{n_3-1}(\bullet)B_+^p(B_+^{n_1-1}(\bullet)B_+^{n_2-1}(\bullet))))).$$

An example

In particular,

$$\begin{aligned}
T(1,1,1;1;1) &= \zeta(J(B_+(\bullet B_+(\bullet\bullet)))) \\
&= \zeta(x(J(\bullet) \sqcup\!\sqcup JB_+(\bullet\bullet))) \\
&= \zeta(x(y \sqcup\!\sqcup x(y \sqcup\!\sqcup y))) \\
&= \zeta(x(y \sqcup\!\sqcup 2xy^2)) \\
&= 2\zeta(x(y \sqcup\!\sqcup xy^2)) \\
&= 2\zeta(x(yxy^2 + 3xy^3)) \\
&= 2\zeta(xyxy^2) + 6\zeta(x^2y^3) \\
&= 2\zeta(x^2yxy) + 6\zeta(x^3y^2) \\
&= 2\zeta(3,2) + 6\zeta(4,1) = \zeta(5),
\end{aligned}$$

where we used an MZV identity at the end.

Extending to rth roots of unity

Another extension of Yamamoto's formalism involves iterated integrals of the forms $\omega_0(t)$ as above, together with

$$\omega_\alpha(t) = \frac{dt}{\alpha^{-1} - t} = \frac{\alpha dt}{1 - \alpha t},$$

where α is an rth root of unity. Here one must replace 2-labeled posets by $(r+1)$-labeled posets, the labels coming from the set $\{0, 1, \epsilon, \dots, \epsilon^{r-1}\}$, where ϵ is a primitive rth root of unity. An $(r+1)$-labeled poset is admissible if minimal elements of X don't have label 0 and maximal elements of X don't have label 1. The we can define $I(X)$ by Eq. (5), and one has Yamamoto's theorem (except that when $r > 1$ one must drop the part about generalized duality).

Nested sums for rth roots of unity

For a chain X, $I(X)$ is a multiple polylogarithm evaluated at rth roots of unity. Specifically, if $X = \{x_1 > x_2 > \cdots > x_k\}$, then

$$I(X) = \int_{1 \geq t_1 > t_2 > \cdots > t_k \geq 0} \omega_{\delta(t_1)}(t_1)\omega_{\delta(t_2)}(t_2) \cdots \omega_{\delta(t_k)}(t_k).$$

Admissibility requires $\delta(t_1) \neq 1$ and $\delta(t_k) \neq 0$, in which case

$$\omega_{\delta(t_1)}\omega_{\delta(t_2)} \cdots \omega_{\delta(t_k)} = (\omega_0)^{i_1 - 1}\omega_{\alpha_1} \cdots (\omega_0)^{i_k - 1}\omega_{\alpha_k},$$

for $\alpha_1, \ldots, \alpha_k \in \{1, \epsilon, \epsilon^2, \ldots, \epsilon^{r-1}\}$ with $\alpha_1 i_1 \neq 1$, and in this case

$$I(X) = \mathrm{Li}_{i_1,\ldots,i_k}(\alpha_1, \frac{\alpha_2}{\alpha_1}, \ldots, \frac{\alpha_k}{\alpha_{k-1}}),$$

where

$$\mathrm{Li}_{i_1,\ldots,i_k}(z_1, \ldots, z_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{z_1^{n_1} \cdots z_k^{n_k}}{n_1^{i_1} \cdots n_k^{i_k}}.$$

The case $r = 2$

The case $r = 1$ is just that treated above. The case $r = 2$ applies to "colored" or alternating MZVs like

$$\zeta(\bar{2}, 1, \bar{3}) = \sum_{l>m>n\geq 1} \frac{(-1)^{l+n}}{l^2 mn^3}$$

and gives nice "diagrammatic" evaluations of them. Shuffle-product formulas for generalized Mordell-Tornheim sums of this type, e.g.,

$$v(p_1, p_2; q) = \sum_{n,m\geq 1} \frac{(-1)^{n+m}}{n^{p_1} m^{p_2} (n+m)^q},$$

(similar to Eq. (6) above) are easily obtained.

The case $r = 2$ cont'd

This produces formulas similar to Eq. (7) above. For example,

$$v(p_1, p_2; q) = \sum_{i=0}^{p_2-1} \binom{p_1 - 1 + i}{i} \zeta(\overline{q + p_1 + i}, p_2 - i) + \sum_{j=0}^{p_1-1} \binom{p_2 - 1 + j}{j} \zeta(\overline{q + p_2 + j}, p_1 - j)$$

and in particular

$$v(2, 1; 1) = \zeta(\bar{2}, 2) + 2\zeta(\bar{3}, 1) = \frac{5}{16}\zeta(4).$$