Using the Implicit Function Theorem

1 The Theorem

Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is a differentiable function, and let $\vec{k} \in \mathbb{R}^m$. Suppose $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$ is a solution to $F(\vec{x}) = \vec{k}$. Write $J^I_F$ for the matrix formed from the first $n-m$ columns of the Jacobian $J_F$, and $J^D_F$ for the matrix formed from the last $m$ columns of $J_F$. If the $m \times m$ matrix $J^D_F(\vec{a})$ is invertible, then there is an open set $U \subset \mathbb{R}^{n-m}$ containing $\vec{b} = \begin{bmatrix} a_1 \\ \vdots \\ a_{n-m} \end{bmatrix}$ and a function $f : U \to \mathbb{R}^m$ so that

$$f(\vec{b}) = \begin{bmatrix} a_{n-m+1} \\ \vdots \\ a_m \end{bmatrix} \quad \text{and} \quad F \begin{bmatrix} \vec{x} \\ f(\vec{x}) \end{bmatrix} = \vec{k} \quad \text{for} \quad \vec{x} \in U.$$

The Jacobian of $f$ is

$$J_f(\vec{x}) = -\left( J^D_F \begin{bmatrix} \vec{x} \\ f(\vec{x}) \end{bmatrix} \right)^{-1} J^I_F \begin{bmatrix} \vec{x} \\ f(\vec{x}) \end{bmatrix} \quad \text{for} \quad \vec{x} \in U.$$

In particular,

$$J_f(\vec{b}) = -(J^D_F(\vec{a}))^{-1} J^I_F(\vec{a}).$$
Two Examples

First Example. Let $F : \mathbb{R}^4 \rightarrow \mathbb{R}$ be given by $F(\vec{x}) = x_1 x_2 - x_3 x_4$, and consider the solution

$$\vec{a} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \end{bmatrix}$$

to the equation $F(\vec{x}) = 7$. Can we regard $x_4$ as a function of $x_1, x_2, x_3$ near this point? In this case $J_F^D$ is the scalar

$$\frac{\partial F}{\partial x_4} = -x_3,$$

so $J_F^D(\vec{a}) = 1$ and indeed $x_4$ is an implicit function of $x_1, x_2, x_3$ near $\vec{a}$. The Jacobian of $x_4 = f(x_1, x_2, x_3)$ is

$$\begin{bmatrix} \frac{\partial x_4}{\partial x_1} & \frac{\partial x_4}{\partial x_2} & \frac{\partial x_4}{\partial x_3} \end{bmatrix} = \frac{1}{x_3} \begin{bmatrix} x_2 & x_1 & -x_3 \end{bmatrix}$$

near this point; in particular, at $\vec{a}$ it is $\begin{bmatrix} -2 & -1 & 5 \end{bmatrix}$.

Second Example. Let $G : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be given by

$$G \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x^2 + xyz + y^2 \\ x^2 + y^2 - z^2 \end{bmatrix},$$

and consider the solution $\vec{a} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ to the equation $G(\vec{x}) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Are $y, z$ functions of $x$ near $\vec{a}$? Here

$$J_G = \begin{bmatrix} 2x + yz & xz + 2y & xy \\ 2x & 2y & -2z \end{bmatrix} \quad \text{and} \quad J_G^D = \begin{bmatrix} xz + 2y & xy \\ 2y & -2z \end{bmatrix}$$

so that

$$J_G^D(\vec{a}) = \begin{bmatrix} 2 & 0 \\ 2 & -4 \end{bmatrix}.$$

This matrix has determinant $-8$, and so is invertible. Thus $y, z$ indeed are functions of $x$ near $\vec{a}$, and their derivatives at $x = 0$ are

$$\begin{bmatrix} \frac{dy}{dx} \\ \frac{dz}{dx} \end{bmatrix} = -\begin{bmatrix} 2 & 0 \\ 2 & -4 \end{bmatrix}^{-1} J_G(\vec{a}) = \begin{bmatrix} -\frac{1}{2} & 0 \\ -\frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -\frac{1}{2} \end{bmatrix}.$$
Hence if $x$ increases by 0.2, then $y$ must decrease by about 0.2 and $z$ must decrease by about 0.1.

### 3 An Economic Model

What follows is an economic model known as the “IS-LM” model; it was developed in 1936 by John R. Hicks to represent ideas of John Maynard Keynes. Let $Y$ be the national income, $C$ national consumption, $I$ national investment, $G$ government expenditure, $M$ the demand for money, $M^s$ the money supply, $T$ taxation, $r$ the interest rate. We assume that $C = f(Y - T)$, $I = I(r)$ and $M = M(Y, r)$ where

$$0 < f'(x) < 1$$

(1) (an increase in the excess of income over tax increases consumption);

$$I'(r) < 0$$

(2) (an increase in the interest rate decreases investment); and

$$\frac{\partial M}{\partial Y} > 0, \frac{\partial M}{\partial r} < 0$$

(3) (monetary demand increases with higher income and decreases with higher interest). In addition, it is assumed that

$$Y = C + I + G \quad \text{and} \quad M^s = M$$

(4) (income is the sum of consumption, investment and government expenditure, and the money supply equals the demand for money). Thus, if we define

$$F = \begin{bmatrix} G \\ M^s \\ T \\ Y \\ r \end{bmatrix} = \begin{bmatrix} Y - C - I - G \\ M - M^s \end{bmatrix},$$

then we are looking for solutions of $F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Now we ask whether $Y, r$ are functions of $G, M^s, T$ subject to the conditions (4). The Jacobian of $F$ is

$$J_F = \begin{bmatrix} -1 & 0 & f'(Y - T) & 1 - f'(Y - T) & -I'(r) \\ 0 & -1 & 0 & \frac{\partial M}{\partial Y} & \frac{\partial M}{\partial r} \end{bmatrix}$$
so

\[ J_F^D = \begin{bmatrix} 1 - f'(Y - T) & -f'(r) \\ \frac{\partial M}{\partial Y} & \frac{\partial M}{\partial r} \end{bmatrix} \]

with determinant

\[ \det J_F^D = (1 - f'(Y - T)) \frac{\partial M}{\partial r} + f'(r) \frac{\partial M}{\partial Y}. \]

Then the assumptions (1,2,3) imply that both terms are negative, so \( \det J_F^D \neq 0 \). So \( Y, r \) can be thought of as functions of \( G, M^*, T \), and

\[
\begin{bmatrix}
\frac{\partial Y}{\partial G} & \frac{\partial Y}{\partial M^*} & \frac{\partial Y}{\partial T}
\end{bmatrix} = -\frac{1}{\det J_F^D} \begin{bmatrix}
\frac{\partial M}{\partial Y} & I'(r) & 0 \\
0 & 1 - f'(Y - T) & 0 \\
0 & 0 & f'(Y - T)
\end{bmatrix}
\]

Thus, for example,

\[ \frac{\partial Y}{\partial G} = -P \frac{\partial M}{\partial r} > 0, \]

where \( P \) is the positive quantity \( -\frac{1}{\det J_F^D} \).

4 Exercises

1. Let \( H : \mathbb{R}^4 \rightarrow \mathbb{R}^2 \) be defined by \( H \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_1^2x_2 + x_3 + x_4^2 \\ x_1x_3x_4 - x_2 \end{bmatrix} \). For each of the following points \( \tilde{a} \), determine whether \( x_3, x_4 \) subject to the condition \( H(\tilde{x}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) are functions of \( x_1, x_2 \) near \( \tilde{a} \). If they are, find \( \left[ \frac{\partial x_3}{\partial x_1} \quad \frac{\partial x_3}{\partial x_2} \quad \frac{\partial x_4}{\partial x_1} \quad \frac{\partial x_4}{\partial x_2} \right] \) at \( \tilde{a} \).

a. \( \tilde{a} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} \)

b. \( \tilde{a} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \)

c. \( \tilde{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \)

d. \( \tilde{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \).

2. In the IS-LM model above, what can you say about \( \frac{\partial Y}{\partial M^*} \) and \( \frac{\partial Y}{\partial T} \)?