Leontief Input-Output Model

We suppose the economy to be divided into \( n \) sectors (about 500 for Leontief’s model). The demand vector \( \vec{d} \in \mathbb{R}^n \) is the vector whose \( i \)th component is the value (in dollars, say) of production of sector \( i \) demanded annually by consumers.

Let \( C = (c_{i,j}) \) be the \( n \times n \) matrix in which \( c_{i,j} \) is the dollar value of the output of sector \( i \) needed by sector \( j \) to produce one dollar of output. If

\[
c_{1,j} + c_{2,j} + \cdots + c_{n,j} < 1,
\]

then it costs sector \( j \) less than one dollar to produce a dollar of output; in that case we say sector \( j \) is profitable. The economy represented by the matrix \( C \) and demand vector \( \vec{d} \) is productive if there is an output vector \( \vec{x} \in \mathbb{R}^n \) so that \( \vec{x} = C\vec{x} + \vec{d} \), or \((I - C)\vec{x} = \vec{d}\). Note that all entries of the demand and output vectors must be nonnegative, and all entries of the matrix \( C \) must also be nonnegative.

Here is a way to think about this. Suppose the sectors first order inputs \( C\vec{d} \) to meet the projected demand \( \vec{d} \). Then they will need to order additional inputs \( C(C\vec{d}) = C^2\vec{d} \) to produce the required \( C\vec{d} \) for each other; this leads to a further required input \( C^3\vec{d} \) to produce \( C^2\vec{d} \), and so forth. The final demand is

\[
\vec{d} + C\vec{d} + C^2\vec{d} + C^3\vec{d} + \cdots = (I + C + C^2 + \cdots)\vec{d}.
\]

If the matrix \( C \) is such that powers \( C^n \) get small as \( n \to \infty \), then the matrix sum \( I + C + C^2 + \cdots \) converges to \((I - C)^{-1}\). This is because

\[
(I + C + C^2 + \cdots + C^n)(I - C) = I - C^{n+1}
\]

for any \( n \), so if \( C^n \to 0 \) as \( n \to \infty \) we can take limits to get

\[
(I + C + C^2 + \cdots)(I - C) = I, \quad \text{or} \quad (I - C)^{-1} = I + C + C^2 + \cdots
\]

In that case we can write the required production vector \( \vec{x} \) as \( \vec{x} = (I - C)^{-1}\vec{d} \). But when does \( C^n \to 0 \) as \( n \to \infty \)? The following result gives an answer.
Theorem 1. If $C$ is a nonnegative matrix (that is, all its entries are nonnegative), then the following conditions are equivalent.

1. $(I - C)^{-1}$ exists and is nonnegative;
2. $C^n \to 0$ as $n \to \infty$;
3. There is a nonnegative vector $\bar{x}$ so that $(I - C)\bar{x}$ is positive (that is, all entries of $(I - C)\bar{x}$ are positive.)

This has the following corollary.

Corollary 1. If the nonnegative matrix $C$ has all its row sums less than 1, then $(I - C)^{-1}$ exists and is nonnegative.

Proof. If $\bar{x} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, note that the $i$th entry of $C\bar{x}$ is just the $i$th column sum of $C$. But then $C\bar{x} < \bar{x}$, so $(I - C)\bar{x} > 0$.

A slight twist to the preceding corollary gives a very natural result.

Corollary 2. If the nonnegative matrix $C$ has all its column sums less than 1, then $(I - C)^{-1}$ exists and is nonnegative.

Proof. Suppose $C$ has all its column sums less than 1. Then by the preceding corollary, $(I - C^T)^{-1}$ exists and is nonnegative. But $I$ is symmetric, so $(I - C^T)^{-1} = ((I - C)^T)^{-1} = ((I - C)^{-1})^T$; and since the transpose of a nonnegative matrix is nonnegative, this means that $(I - C)^{-1}$ exists and is nonnegative.

Now the $j$th column sum of a consumption matrix being less than 1 says that sector $j$ is profitable, so this latter corollary can be stated as follows.

Corollary 3. An economy is productive for any demand vector $\bar{d} \geq 0$ if each sector is profitable.

Call a consumption matrix $C$ productive if the economy represented by $C$ and any demand vector $\bar{d} \geq 0$ is productive. While Theorem 1 gives a necessary and sufficient condition for $C$ to be productive, the condition that
there is some nonnegative vector $\bar{x}$ making $(I - C)\bar{x}$ positive, or equivalently $C\bar{x} < \bar{x}$, is not easy to check. (The hypotheses of Corollaries 1 and 2 are easy to check, but they may not be true.) So other criteria have been found. The following sufficient condition is called the Hawkins-Simon condition in the economics literature.

**Theorem 2.** If $C$ is nonnegative and every principal minor of $I - C$ is positive, then $(I - C)^{-1}$ is nonnegative.

Another criterion can be given involving the eigenvalues of $C$. The Perron-Frobenius theorem states that if $C$ is a nonnegative matrix, then there is a real eigenvalue $\lambda_{pf}$ of $C$ such that $\lambda_{pf} \geq 0$ and $|\lambda| < \lambda_{pf}$ for all other eigenvalues $\lambda$ of $C$. We call $\lambda_{pf}$ the maximal eigenvalue of $C$. Then the following result holds.

**Theorem 3.** A nonnegative matrix $C$ is productive if and only if the maximal eigenvalue $\lambda_{pf}$ of $C$ satisfies $\lambda_{pf} < 1$.

To illustrate these three theorems, we shall take some examples of three-sector economies. Remember that this is just for illustration, since any realistic input-output model has hundreds of sectors. First consider the economy with consumption matrix

$$C = \begin{bmatrix} 0 & .65 & .55 \\ .25 & .05 & .1 \\ .25 & .05 & 0 \end{bmatrix}.$$ 

Then $C$ satisfies the hypothesis of Corollary 2, since every sector is profitable. But the hypothesis of Corollary 1 doesn’t hold, since the first row sum is 1.2. Since

$$I - C = \begin{bmatrix} 1 & -.65 & -.55 \\ -.25 & .95 & -.1 \\ -.25 & -.05 & 1 \end{bmatrix},$$ 

has principal minors 1, 0.7875, 0.62875, the Hawkins-Simon condition is satisfied. The hypothesis of Theorem 3 also holds in this case, since $C$ has maximal eigenvalue $\lambda_{pf} \approx 0.60174$.

On the other hand, the three-sector economy with consumption matrix

$$C = \begin{bmatrix} .5 & .15 & 0 \\ .25 & .15 & .5 \\ .25 & .15 & .5 \end{bmatrix}$$

...
fails to satisfy the hypothesis of Corollary 2; indeed sectors 1 and 3 are not profitable. But it does satisfy the hypothesis of Corollary 1, since the row sums are 0.65, 0.9, and 0.9. The hypothesis of Theorem 2 also holds, since

$$I - C = \begin{bmatrix} 0.5 & -0.15 & 0 \\ -0.25 & 0.85 & -0.5 \\ -0.25 & -0.15 & 0.5 \end{bmatrix}$$

has principal minors 0.5, 0.3875, 0.1375. In addition, $C$ has maximal eigenvalue $\lambda_{pf} \approx 0.78267$, so the hypothesis of Theorem 3 is satisfied as well.

Finally, consider

$$C = \begin{bmatrix} 0.7 & 0.3 & 0.2 \\ 0.1 & 0.4 & 0.3 \\ 0.2 & 0.4 & 0.1 \end{bmatrix}.$$ 

This matrix violates the hypothesis of Corollary 2, since sectors 1 and 2 are not profitable. It also doesn’t satisfy the hypothesis of Corollary 1, since the first row sum is 1.2. With some persistence Theorem 1 can still be used to show $C$ productive, since

$$C \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.9 \\ 0.9 \\ 0.9 \end{bmatrix} < \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$ 

The principal minors of

$$I - C = \begin{bmatrix} 0.3 & -0.3 & -0.2 \\ -0.1 & 0.6 & -0.3 \\ -0.2 & -0.4 & 0.9 \end{bmatrix}$$

are 0.3, 0.15, 0.049, so Theorem 2 implies that $C$ is productive. Also, $C$ has maximal eigenvalue $\lambda_{pf} \approx 0.92736$, so Theorem 3 applies as well.

But how can we show a consumption matrix is not productive? One way is by Theorem 3, but this requires knowing the maximal eigenvalue of $C$. We can at least get some information from row and column sums. As we’ve seen, $C$ may still be productive even though individual row sums or column sums exceed 1. But what if, say, all the column sums of $C$ are 1 or more? If $C$ is productive, Theorem 1 implies that there is a vector $\vec{x} \geq \vec{0}$ with

$$C\vec{x} = \begin{bmatrix} c_{1,1}x_1 + c_{1,2}x_2 + \cdots + c_{1,n}x_n \\ c_{2,1}x_1 + c_{2,2}x_2 + \cdots + c_{2,n}x_n \\ \vdots \\ c_{n,1}x_1 + c_{n,2}x_2 + \cdots + c_{n,n}x_n \end{bmatrix} < \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$ 

4
Adding these $n$ inequalities gives

$$(c_{1,1} + c_{2,1} + \cdots + c_{n,1})x_1 + (c_{1,2} + c_{2,2} + \cdots + c_{n,2})x_2 + \cdots + (c_{1,n} + c_{2,n} + \cdots + c_{n,n})x_n < x_1 + x_2 + \cdots + x_n.$$ But we’ve assumed that all the column sums $c_{1,1} + \cdots + c_{n,1}$, $c_{1,2} + \cdots + c_{n,2}$, ..., $c_{1,n} + \cdots + c_{n,n}$ are $\geq 1$, so we have the contradictory inequality

$$(c_{1,1} + c_{2,1} + \cdots + c_{n,1})x_1 + (c_{1,2} + c_{2,2} + \cdots + c_{n,2})x_2 + \cdots + (c_{1,n} + c_{2,n} + \cdots + c_{n,n})x_n \geq x_1 + x_2 + \cdots + x_n.$$ This contradiction shows that $C$ cannot be productive, giving us another corollary to Theorem 1.

**Corollary 4.** The nonnegative matrix $C$ cannot be productive if every column sum is 1 or more.

To put it another way, an economy cannot be productive if every sector is not profitable. By taking transposes as above we also have the following result.

**Corollary 5.** The nonnegative matrix $C$ cannot be productive if every row sum is 1 or more.

**Exercises**

1. Determine which of the following consumption matrices is productive.
   
   a. $\begin{bmatrix} .5 & .4 & .2 \\ .2 & .3 & .3 \\ .3 & .4 & .4 \end{bmatrix}$
   
   b. $\begin{bmatrix} .7 & .3 & .25 \\ .2 & .4 & .25 \\ .05 & .15 & .25 \end{bmatrix}$
   
   c. $\begin{bmatrix} .1 & .5 & .4 \\ .4 & .3 & .3 \\ .3 & .4 & .4 \end{bmatrix}$
   
   d. $\begin{bmatrix} .1 & .1 & .3 \\ .9 & .8 & .3 \\ .1 & .5 & .4 \end{bmatrix}$

2. Give an example of a (nonzero) productive consumption matrix that is not invertible.
3. For real numbers $t \geq 0$, let $C(t) = \begin{bmatrix} .7 & .3 & .2 \\ .1 & .4 & .3 \\ .2 & .4 & t \end{bmatrix}$. For which $t$’s can you say whether $C(t)$ is productive or not?