An Odd Variant of Multiple Zeta Values

Michael E. Hoffman

U. S. Naval Academy

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Outline

1. Introduction
2. MtVs as images of quasi-symmetric functions
3. Iterated integrals
4. Calculations and conjectures
5. Depth 2 results
Introduction

For positive integers $a_1, \ldots, a_k$ with $a_1 > 1$ we define the corresponding multiple $t$-value (MtV) by

$$t(a_1, a_2, \ldots, a_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1, \ n_i \text{ odd}} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_k^{a_k}}.$$ 

Except for the specification that the $n_i$ are odd, this is just the usual definition of the multiple zeta value $\zeta(a_1, \ldots, a_k)$. We say $t(a_1, \ldots, a_k)$ has depth $k$ and weight $a_1 + \cdots + a_k$. MtVs of depth 1 are simply related to values of the Riemann zeta function:

$$t(a) = \sum_{n > 0 \text{ odd}} \frac{1}{n^a} = \zeta(a) - \sum_{n > 0 \text{ even}} \frac{1}{n^a} = (1 - 2^{-a}) \zeta(a). \quad (1)$$
While multiple zeta values (for depth $\leq 2$) have a history going back to Euler, $t$-values can only claim a history of 98 years. In 1906 N. Nielsen showed that

$$\sum_{i=1}^{n-1} t(2i)t(2n-2i) = \frac{2n - 1}{2},$$

which parallels the result Euler gave for zeta values:

$$\sum_{i=1}^{n-1} \zeta(2i)\zeta(2n-2i) = \frac{2n + 1}{2}. $$
It turns out that the theory of multiple $t$-values is in some ways directly parallel to that for multiple zeta values, and in other ways completely different. The MtVs share with the MZVs the “harmonic algebra” (AKA “stuffle”) multiplication, e.g., just as

$$\zeta(2)\zeta(3, 1) = \zeta(2, 3, 1) + \zeta(3, 2, 1) + \zeta(3, 1, 2) + \zeta(5, 1) + \zeta(3, 3)$$

we have

$$t(2)t(3, 1) = t(2, 3, 1) + t(3, 2, 1) + t(3, 1, 2) + t(5, 1) + t(3, 3).$$

From this (and equation (1)) follows, e.g.,

$$t\underbrace{(2, \ldots, 2)}_n = \frac{\pi^{2n}}{2(2n)!}, \quad \text{cf.} \quad \zeta\underbrace{(2, \ldots, 2)}_n = \frac{\pi^{2n}}{(2n + 1)!}.$$
Where MZVs and MtVs differ radically is in their representation as iterated integrals. The iterated integral representation of MZVs leads to a second algebra structure (shuffle product) and the duality theorem, but the iterated integral representation of MtVs is not nearly so nice. This is already apparent in weight 3: while for MZVs one has the first instance of duality

\[ \zeta(2, 1) = \zeta(3) \]

(discovered by Euler and many others since), for MtVs one has

\[ t(2, 1) = -\frac{1}{2} t(3) + t(2) \log 2. \]
Nevertheless, the iterated integral representation for MtVs does lead to a formula expressing any MtV as a sum of alternating MZVs. Thanks to the Multiple Zeta Value Data Mine project of Blümlein, Broadhurst and Vermaseren, extensive tables expressing alternating MZVs in terms of a (provisional) basis exist. Our calculations using these tables have led us to the following conjecture.

**Conjecture**

*The dimension of the rational vector space spanned by MtVs of weight $n \geq 2$ is $F_n$, the $n$th Fibonacci number.*

The corresponding conjecture for MZVs reads the same, but with $F_n$ replaced by $P_n$, the $n$th Padovan number (i.e., $P_0 = 1$, $P_1 = 0$, $P_2 = 1$, $P_n = P_{n-2} + P_{n-3}$).
MtVs as Multiple Hurwitz Zeta Functions

In the literature there are several results about multiple $t$-values, but these are usually discussed in the context of multiple Hurwitz zeta values, defined by

$$\zeta(a_1, \ldots, a_k; p_1, \ldots, p_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{(n_1 + p_1)^{a_1} \cdots (n_k + p_k)^{a_k}}.$$  

It is evident that

$$t(a_1, \ldots, a_k) = 2^{-a_1 - \cdots - a_k} \zeta(a_1, \ldots, a_k; -\frac{1}{2}, \ldots, -\frac{1}{2}).$$
Quasi-Symmetric Functions

Consider the algebra \( \mathbb{Q}[[x_1, x_2, \ldots]] \) of formal power series in a countable set of commuting generators \( x_1, x_2, \ldots \). This has a graded subalgebra \( \mathfrak{P} \) consisting of those series of bounded degree (where each \( x_i \) has degree 1). An element of \( f \in \mathfrak{P} \) is a quasi-symmetric function if

\[
\text{coefficient of } x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k} \text{ in } f = \text{coefficient of } x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} \text{ in } f
\]

whenever \( i_1 < i_2 < \cdots < i_k \). The set of such \( f \) is a subalgebra \( \mathbb{Q}\text{Sym} \) of \( \mathfrak{P} \), called the quasi-symmetric functions.
Quasi-Symmetric Functions cont’d

As a vector space, QSym has the basis consisting of monomial quasi-symmetric functions $M(a_1,\ldots,a_k)$, where $(a_1,\ldots,a_k)$ is a composition (ordered sequence) of positive integers and

$$M(a_1,\ldots,a_k) = \sum_{i_1<\cdots<i_k} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}.$$ 

QSym has as a subalgebra the symmetric functions Sym; the symmetric functions have the vector space basis

$$m_{a_1,\ldots,a_k} = \sum_{i_1,\ldots,i_k \text{ distinct}} x_{i_1}^{a_1} \cdots x_{i_k}^{a_k}$$

indexed by partitions (unordered sequences) of positive integers. Note that, e.g.,

$$m_{2,2,1} = M_{(2,2,1)} + M_{(2,1,2)} + M_{(1,2,2)}.$$
In fact, QSym is a polynomial algebra on certain of the $M_I$. We order compositions lexicographically, i.e,

$$(1) < (1, 1) < (1, 2) < (1, 2, 1) < \cdots$$

and call a composition $I$ Lyndon if $I < K$ for any proper factorization $I = JK$: e.g., $(1, 3)$ is Lyndon but $(1, 1)$ and $(1, 2, 1)$ aren’t. Then QSym is the polynomial algebra on the $M_I$ with $I$ Lyndon (Malvenuto and Reutenauer, 1995). Note that the only Lyndon composition ending in 1 is $(1)$. Let $\text{QSym}^0$ be the subalgebra of QSym generated by $M_I$ for $I \neq (1)$ Lyndon, so that QSym $= \text{QSym}^0[M_{(1)}]$. 
The following result is well-known (Hoffman 1997).

**Theorem**

There is a homomorphism \( \zeta : Q\text{Sym}^0 \rightarrow \mathbb{R} \) sending 1 to 1 and \( M_{(a_1,a_2,\ldots,a_k)} \) to \( \zeta(a_k, \ldots, a_2, a_1) \) for \( a_k \geq 2 \).

This is paralleled by

**Theorem**

There is a homomorphism \( \theta : Q\text{Sym}^0 \rightarrow \mathbb{R} \) sending 1 to 1 and \( M_{(a_1,a_2,\ldots,a_k)} \) to \( t(a_k, \ldots, a_2, a_1) \) for \( a_k \geq 2 \).

It follows that if any linear combination of \( t \)-values is the image of a symmetric function, then it is actually expressible as a rational polynomial in the depth 1 \( t \)-values, e.g.,

\[
t(2, 3) + t(3, 2) = t(2)t(3) - t(5).
\]
In particular, any $t$-value of repeated arguments is a rational polynomial in the $t(i)$. If we write $\{k\}_n$ for $k$ repeated $n$ times, then we have the following.

**Theorem**

For integer $k \geq 2$, let $Z_k(x)$ be the generating function

$$Z_k(x) = 1 + \sum_{i=1}^{\infty} \zeta(\{k\}_i)x^{ik}.$$  

Then

$$1 + \sum_{i=1}^{\infty} t(\{k\}_i)x^{ik} = \frac{Z_k(x)}{Z_k(\frac{x}{2})}.$$
Now the generating functions $Z_k(x)$ were studied in some detail back in the 1990’s: see Broadhurst, Borwein and Bradley 1997. From their results and the preceding theorem we have, e.g.,

\[ t(\{4\}_n) = \frac{\pi^{4n}}{2^{2n}(4n)!}, \quad t(\{6\}_n) = \frac{\pi^{6n}}{(8n)(6n - 1)!}, \]

\[ t(\{10\}_n) = \frac{\pi^{10n}(L_{10n} + 1)}{(32n)(10n - 1)!}, \]

where in the last formula $L_n$ is the $n$th Lucas number, i.e., $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$. 
Iterated Integral Representations

Call a sequence of \((a_1, \ldots, a_k)\) of positive integers admissible if \(a_1 > 1\). Now if

\[
\omega_0 = \frac{dt}{t}, \quad \omega_1 = \frac{dt}{1 - t}, \quad \omega_{-1} = \frac{dt}{1 + t},
\]

then there is the well-known representation

\[
\zeta(a_1, \ldots, a_k) = \int_0^1 \omega_0^{a_1-1} \omega_1 \cdots \omega_0^{a_k-1} \omega_1
\]

for any admissible sequence \((a_1, \ldots, a_k)\), which extends as follows. If \((a_1, \ldots, a_k)\) is admissible and \(\epsilon_1, \ldots, \epsilon_k \in \{-1, 1\}\), then

\[
\int_0^1 \omega_0^{a_1-1} \omega_{\epsilon_1} \cdots \omega_0^{a_k-1} \omega_{\epsilon_k} = \\
\epsilon_1 \cdots \epsilon_k \zeta(\epsilon_1 a_1, \epsilon_1 \epsilon_2 a_2, \epsilon_2 \epsilon_3 a_3, \ldots, \epsilon_{k-1} \epsilon_k a_k)
\]
where for $a_i \in \{-1, 1\}$ we define the alternating MZV by

$$\zeta(a_1, \ldots, a_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{\text{sgn}(a_1)^{n_1} \cdots \text{sgn}(a_k)^{n_k}}{n_1^{a_1} \cdots n_k^{a_k}}.$$ 

For multiple $t$-values there is also an iterated integral representation, but it is not so simple.

**Proposition**

*For any admissible sequence $(a_1, \ldots, a_k)$,*

$$t(a_1, \ldots, a_k) = \int_0^1 \omega_0^{a_1-1} \Omega_a \cdots \omega_0^{a_k-1} \Omega_a \omega_0^{a_k-1} \Omega_b$$

*where*

$$\Omega_a = \frac{tdt}{1 - t^2} = \frac{1}{2}(\omega_1 - \omega_{-1}), \quad \Omega_b = \frac{dt}{1 - t^2} = \frac{1}{2}(\omega_1 + \omega_{-1}).$$
Expanding out the preceding result gives

**Theorem**

For an admissible sequence \((a_1, \ldots, a_k)\), the multiple t-value \(t(a_1, \ldots, a_k)\) is given by

\[
\frac{1}{2^k} \sum_{\epsilon_1, \ldots, \epsilon_k} \epsilon_1 \cdots \epsilon_k \zeta(\epsilon_1 a_1, \ldots, \epsilon_k a_k),
\]

where the sum is over the \(2^k\) k-tuples \((\epsilon_1, \ldots, \epsilon_k)\) with \(\epsilon_i \in \{-1, 1\}\).
Using the preceding result, we can expand any MtV into a sum of alternating MZVs. For example,

\[ t(2, 3, 1) = \frac{1}{8} \left[ \zeta(2, 3, 1) - \zeta(\bar{2}, 3, 1) - \zeta(2, \bar{3}, 1) - \zeta(2, 3, \bar{1}) \\ + \zeta(\bar{2}, \bar{3}, 1) + \zeta(\bar{2}, 3, \bar{1}) + \zeta(2, \bar{3}, \bar{1}) - \zeta(\bar{2}, \bar{3}, \bar{1}) \right], \]

where we have used the usual convention of writing \( \bar{n} \) instead of \(-n\) in the exponent string. Using results of the Multiple Zeta Values Data Mine project of Blümlein, Broadhurst and Vermaseren, we expand each of the alternating MZVs in this sum to obtain

\[ t(2, 3, 1) = -\frac{2}{21} t(6) - \frac{3}{196} t(3)^2 - \frac{1}{2} t(2)\zeta(\bar{3}, 1) + \frac{1}{4} \zeta(\bar{5}, 1) \]
\[ - \frac{1}{2} t(5) \log 2 + \frac{4}{7} t(2) t(3) \log 2. \]
Similarly we have obtained formulas for all multiple $t$-values of weight $\leq 7$. In our tables we have

$$t(2, 3) = -\frac{1}{2} t(5) + \frac{4}{7} t(2) t(3).$$

This and many similar instances have led us to the following conjecture.

**Conjecture**

*Every multiple $t$-value admits a representation as a polynomial in the $t(i)$, $i \geq 2$, log 2, and selected alternating MZVs in such a way that*

$$\frac{\partial t(s, 1)}{\partial \log 2} = t(s).$$

*for any admissible sequence $s$.***
The Fibonacci Conjecture

The second conjecture is based on computations of the rank of the set of multiple $t$-values of a given weight. Based on our tables through weight 7, we have computed the ranks (starting in weight 2) as

$$1, 2, 3, 5, 8, 13$$

which leads to the obvious conjecture

Conjecture

The dimension of the rational vector space spanned by weight-$n$ multiple $t$-values is $F_n$. 
Of course the existing evidence for the Fibonacci Conjecture is rather thin. The Multiple Zeta Value Data Mine actually has publicly available files on alternating MZVs for weights through 14, so there is plenty of testing yet to do even without extending their work. But this gets laborious: in weight 7 there are 32 admissible sequences, and the sum for $t(2,1,1,1,1,1,1)$ has 64 terms. Blümlein, Broadhurst and Vermaseren are all particle physicists, with plenty of experience with large-scale calculations, so they probably have helpful ideas on how to manage the higher weights.
Explicit Formulas in Depth 2

One place where the theories of MZVs and MtVs are closely parallel is in depth 2. It is convenient to define

$$N(a, b, i) = \binom{2i - 2}{a - 1} + \binom{2i - 2}{b - 1},$$

where $\binom{p}{j} = 0$ if $j > p$. It is well-known (and implicit in Euler’s 1775 paper) that

$$\zeta(a, b) = \frac{(-1)^a}{2} \left( \binom{a + b}{a} - 1 \right) \zeta(2n + 1) +$$

$$\begin{cases} 
\zeta(a)\zeta(b) - \sum_{i=2}^{n} N(a, b, i)\zeta(2i - 1)\zeta(2n - 2i + 2), & a \text{ even,} \\
\sum_{i=2}^{n} N(a, b, i)\zeta(2i - 1)\zeta(2n - 2i + 2), & \text{otherwise.}
\end{cases}$$

if $a + b = 2n + 1$ and $a, b \geq 2$ (No general formula in terms of single zeta values is known for $\zeta(a, b)$ if $a + b$ is even.)
Explicit Formulas in Depth 2 cont’d

For $\zeta(2n, 1)$ one has another formula

$$\zeta(2n, 1) = n\zeta(2n + 1) - \frac{1}{2} \sum_{i=2}^{2n-1} \zeta(i)\zeta(2n + 1 - i).$$

Basu 2008 gave the similar formula

$$t(a, b) = -\frac{1}{2} t(2n + 1) +$$

$$\begin{cases} 
    t(a)t(b) - \sum_{i=2}^{n} N(a, b, i) \frac{t(2i-2)t(2n-2i+2)}{2^{2i-1} - 1}, & a \text{ even}, \\
    \sum_{i=2}^{n} N(a, b, i) \frac{t(2i-1)t(2n-2i+2)}{2^{2i-1} - 1}, & \text{otherwise},
\end{cases}$$

for $a + b = 2n + 1$. Basu actually stipulates that $a, b \geq 2$, but if we take $a = 2n, b = 1$ and set $t(1) = \log 2$ the formula is still valid (as follows from Nakamura and Tasaka 2013).
Explicit Formulas in Depth 2 cont’d

The proofs used by Basu and Nakamura-Tasaka are elementary but different. I have been able to prove the \(a = 2n, b = 1\) case, that is,

\[
t(2n, 1) = -\frac{1}{2} t(2n + 1) + t(2n) \log 2
\]

\[
- \sum_{i=2}^{n} \frac{t(2i - 1)t(2n - 2i + 2)}{2^{2i-1} - 1}
\]

using yet another elementary method. I believe this can be extended to the general case \(a + b = 2n + 1\), though I haven’t yet done this. My method involves modified Tornheim sums.
The depth 2 formulas for the MZV case can all be rather neatly proved using sums invented in Tornheim 1950:

\[ T(a, b, c) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{ia^j b(i + j)^c}. \]

They have the properties \( T(a, b, c) = T(b, a, c) \), \( T(a, 0, c) = \zeta(c, a) \) for \( c > 1 \), \( T(a, b, 0) = \zeta(a)\zeta(b) \) for \( a, b \geq 2 \), and

\[ T(a, b, c) = T(a - 1, b, c + 1) + T(a, b - 1, c + 1) \]

for \( a, b \geq 1 \).
For the $t$-values one can use a modified Tornheim sum

$$M(a, b, c) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(2i)^a(2j + 1)^b(2i + 2j + 1)^c}$$

One has $M(0, b, c) = t(c, b)$ for $c > 1$,  
$M(a, b, 0) = 2^{-a} \zeta(a)t(b)$ for $a, b \geq 2$ and 

$$M(a, b, c) = M(a - 1, b, c + 1) + M(a, b - 1, c + 1)$$

for $a, b \geq 1$, but the symmetry in the first two arguments is lost and $M(a, 0, c)$ is not particularly simple. Nevertheless, one can mimic many of the arguments used for Tornheim sums.
Tornheim and Modified Tornheim Sums

Here is an interesting point of difference between $T$ and $M$. By partial fractions one has

$$T(1, p, 1) = \sum_{j=1}^{\infty} \frac{1}{j^{p+1}} \sum_{i=1}^{\infty} \left[ \frac{1}{i} - \frac{1}{i+j} \right],$$

and the inner sum telescopes to give

$$\sum_{j=1}^{\infty} \frac{1}{j^{p+1}} \left[ 1 + \cdots + \frac{1}{i} \right] = \zeta(p + 2) + \zeta(p + 1, 1).$$
But in the corresponding sum

\[ M(1, p, 1) = \sum_{j=0}^{\infty} \frac{1}{(2j + 1)^{p+1}} \sum_{i=1}^{\infty} \left[ \frac{1}{2i} - \frac{1}{2i + 2j + 1} \right] \]

the inner sum does not telescope, and instead we have

\[ \sum_{j=0}^{\infty} \frac{1}{(2j + 1)^{p+1}} \left[ -\log 2 + 1 + \frac{1}{3} + \cdots + \frac{1}{2j + 1} \right] \]

\[ = t(p + 2) + t(p + 1, 1) - t(p + 1) \log 2. \]
A Formula for $t(2n + 1, 1)$

The argument for $t(2n, 1)$ can be modified slightly to give a formula for $t(2n + 1, 1)$ (but one must pay the price of including an alternating MZV):

$$t(2n + 1, 1) = t(2n + 1) \log 2 - \frac{1}{2} \zeta(2n + 1, 1) +$$

$$\frac{(2n + 1)(1 - 2^{2n+1}) - 2^{2n+2}}{4(2^{2n+2} - 1)} t(2n + 2) +$$

$$\sum_{i=1}^{n-1} \frac{2^{2n+1} - 2^{2i+1} - 2^{2n-2i+1} + 1}{2(2^{2i+1} - 1)(2^{2n-2i+1} - 1)} t(2i + 1) t(2n - 2i + 1).$$