Sum Theorems for Multiple Zeta Values, Old and New

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Outline

1. Introduction
2. Classical Sum, Duality and Derivation Theorems
3. Multiple Zeta-Star Values
4. An Algebraic Framework
5. Yamamoto’s Product
6. New Sum Theorems?
For positive integers $a_1, \ldots, a_k$ with $a_1 > 1$ we define the corresponding multiple zeta value (MZV) by

$$
\zeta(a_1, a_2, \ldots, a_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_k^{a_k}}. \tag{1}
$$

One calls $k$ the depth and $a_1 + \cdots + a_k$ the weight. Euler already studied the cases of depth 1 and depth 2, but arguably the present era of MZVs of general depth began with the proof of the "sum theorem"

$$
\sum_{a_1 + \cdots + a_k = n, \ a_1 > 1, \ a_i \geq 1} \zeta(a_1, \ldots, a_k) = \zeta(n). \tag{2}
$$

This was proved by Euler for depth 2, by Courtney Moen for depth 3, and by Andrew Granville and Don Zagier for general depth.
In 1997 I introduced the algebraic notation which has since become fairly standard. Let $\mathcal{H}$ be the underlying rational vector space of the noncommutative polynomial algebra $\mathbb{Q}\langle x, y \rangle$, graded by giving both $x$ and $y$ degree 1. Also, let $\mathcal{H}^1 = \mathbb{Q}1 + \mathcal{H}y$ and $\mathcal{H}^0 = \mathbb{Q}1 + x\mathcal{H}y \subset \mathcal{H}^1$. Then we can think of MZVs as images under the $\mathbb{Q}$-linear map $\zeta : \mathcal{H}^0 \to \mathbb{R}$ that sends the empty word $1 \in \mathcal{H}^0$ to $1 \in \mathbb{R}$, and the word $x^{a_1-1}y \cdots x^{a_k-1}y$ to $\zeta(a_1, \ldots, a_k)$. We sometimes think of $\mathcal{H}^1$ as $\mathbb{Q}\langle z_1, z_2, \ldots \rangle$, where $z_i = x^{i-1}y$ has degree $i$. Then $\mathcal{H}^0 \subset \mathcal{H}^1$ is the subspace generated by 1 and all words that don’t begin with $z_1$. 
The main point of my 1997 paper was that one can make \( \mathcal{H}_1 \) a commutative algebra by giving it the product \( \ast \) defined inductively by \( w \ast 1 = 1 \ast w = w \) for all words \( w \) of \( \mathcal{H}_1 \), and

\[
z_i u \ast z_j v = z_i(u \ast z_j v) + z_j(z_i u \ast v) + z_{i+j}(u \ast v)
\]

for all words \( u, v \) of \( \mathcal{H}_1 \). Then one has the following results.

**Theorem**

The graded algebra \( (\mathcal{H}_1, \ast) \) is isomorphic to the algebra \( \text{QSym} \) of quasi-symmetric functions, which is polynomial on Lyndon words.

**Theorem**

The \( \mathbb{Q} \)-linear function \( \zeta : \mathcal{H}_0 \to \mathbb{R} \) is a homomorphism of algebras from \( (\mathcal{H}_0, \ast) \) to \( \mathbb{R} \).
In the algebraic notation, the classical sum theorem (2) says that $\zeta$ sends the sum of all words in $H^0$ of degree $n$ and $y$-degree $k < n$ to $\zeta(n)$, regardless of $k$. Two further striking results for MZVs lend themselves to the algebraic notation. Let $\tau$ be the antiautomorphism of $\mathbb{Q}\langle x, y \rangle$ that exchanges $x$ and $y$ (so, e.g., $\tau(xy^2y) = xy^2xy$), and let $D$ be the derivation of $\mathbb{Q}\langle x, y \rangle$ that sends $x$ to 0 and $y$ to $xy$. Then we have the following results.

**Theorem**

For all words $w$ of $H^0$, $\zeta(\tau(w)) = \zeta(w)$.

**Theorem**

For all words $w$ of $H^0$, $\zeta(D(w)) = \zeta(\tau D\tau(w))$. 
Cyclic Sum Theorem

There is another derivation theorem that is a bit more mysterious. Regard $\mathcal{H} \otimes \mathcal{H}$ as a two-sided module over $\mathcal{H}$ via

$$a(b \otimes c) = ab \otimes c \quad \text{and} \quad (a \otimes b)c = a \otimes bc.$$ 

Now let $\hat{\mathcal{C}} : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ be the derivation sending $x$ to 0 and $y$ to $y \otimes x$, and let $\overline{\mu}(a \otimes b) = ba$. Then we define the “cyclic derivation” $\overline{\mu} \hat{\mathcal{C}}$, so, e.g, $C(z_4 z_2) = C(x^3 y x y)$ is

$$\overline{\mu} \hat{\mathcal{C}}(x^3 y x y) = \overline{\mu}(x^3 y \otimes x^2 y + x^3 y x y \otimes x) = x^2 y x^3 y + x^4 y x y$$

$$= z_3 z_4 + z_5 z_2.$$

Then the cyclic sum theorem, conjectured by me and proved by Yasuo Ohno, reads as follows.
For any word of $\mathbb{H}^1$ not a power of $y$, $\zeta(C(w)) = \zeta(\tau C \tau(w))$.

For example, for $w = z_4z_2$ it says that

$\zeta(5, 2) + \zeta(3, 4) = \zeta(4, 2, 1) + \zeta(3, 2, 2) + \zeta(2, 2, 3) + \zeta(2, 4, 1)$

while the derivation theorem applied to the same word is

$\zeta(5, 2) + \zeta(4, 3) = \zeta(4, 1, 2) + \zeta(3, 2, 2) + \zeta(2, 3, 2) + \zeta(4, 2, 1)$. 

Ohno’s proof (like mine for the derivation theorem) uses partial fractions. Interestingly, the cyclic sum theorem implies the sum theorem (which is not the case for the derivation theorem, except for depth $\leq 3$).
Euler actually worked in terms of what are now called multiple zeta-star values (MZSVs), i.e.,

$$
\zeta^*(a_1, a_2, \ldots, a_k) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_k^{a_k}}. \quad (3)
$$

The difference from eqn. (1) is that $\geq$ replaces $>$. Of course $\zeta^*(n) = \zeta(n)$. One has, e.g.,

$$
\zeta^*(3, 2, 1) = \zeta(3, 2, 1) + \zeta(5, 1) + \zeta(3, 3) + \zeta(6). \quad (4)
$$

The MZSVs give the same rational vector space as the MZVs, since equations like (4) can be inverted:

$$
\zeta(3, 2, 1) = \zeta^*(3, 2, 1) - \zeta^*(5, 1) - \zeta^*(3, 3) + \zeta^*(6).
$$
In my 1992 paper I actually treated the MZVs and MZSVs on a fairly equal footing (denoting $\zeta(a_1, \ldots, a_k)$ by $A(a_1, \ldots, a_k)$ and $\zeta^*(a_1, \ldots, a_k)$ by $S(a_1, \ldots, a_k)$). I noted that the sum theorem (2) is equivalent to the statement that summing $\zeta^*(a_1, \ldots, a_k)$ over all strings with $a_1 > 1$ and $a_1 + \cdots + a_k = n$ gives

$$\binom{n-1}{k-1} \zeta(n).$$

But one loses the duality and derivation theorems. I didn’t mention the zeta-star values in my 1997 paper, but I easily could have, since the algebra of MZSVs parallels that for MZVs.
Instead of defining the product $\ast$ on $\mathcal{H}^1$ as above, one can define an algebra $\star$ on $\mathcal{H}^1$ inductively by $w \star 1 = 1 \ast w = w$ for all words $w$ of $\mathcal{H}^1$, and

$$z_i u \star z_j v = z_i (u \ast z_j v) + z_j (z_i u \ast v) - z_{i+j} (u \ast v)$$

for all $i, j \geq 1$ and words $u, v$ of $\mathcal{H}^1$. Then the $\mathbb{Q}$-linear map $\zeta^\ast : \mathcal{H}^0 \to \mathbb{R}$ sending $z_{a_1} \cdots z_{a_k}$, $a_1 > 1$, to $\zeta^\ast (a_1, \ldots, a_k)$ is a homomorphism from $(\mathcal{H}^0, \ast)$ to $\mathbb{R}$. Also, $(\mathcal{H}^1, \star)$ is isomorphic as a graded rational algebra to $(\mathcal{H}^1, \ast)$. From this point of view MZVs and MZSVs look rather similar, but (despite the loss of the duality theorem) there are some advantages to MZSVs.
Cyclic Sum Theorem for MZSVs

For example, despite the lack of an obvious analogue of the derivation theorem for MZSVs, there is a cyclic sum theorem for MZSVs that is actually simpler than the one for MZVs. This result is due to Ohno and Noriko Wakabayashi and can be stated as follows.

**Theorem (Ohno-Wakabayashi)**

For all words $w$ of $\mathfrak{S}_1$ not a power of $y$,

$$\zeta^*(\tau C\tau(w)) = n\zeta(n + 1), \text{ where } n = \deg w.$$

Of course this implies the sum theorem.
Let’s go back to the vector space $\mathcal{H}^1$, which we now know has two (isomorphic) commutative algebra structures $(\mathcal{H}^1, \ast)$ and $(\mathcal{H}^1, \star)$. There’s actually some more structure here, which I started to develop in my 2000 paper “Quasi-shuffle products” and developed further in joint work here in 2012 with Kentaro Ihara. Define a product $\circ$ on the $z_i$ by $z_i \circ z_j = z_{i+j}$. Suppose $w$ is a word of $\mathcal{H}^1$, say $w = a_1 \cdots a_k$, where each $a_j$ is a letter $z_i$. Given a composition $I = (i_1, \ldots, i_l)$ of $k$ (i.e., the $i_j$ are positive integers whose sum is $k$), define $I[w]$ to be the word

$$(a_1 \circ \cdots \circ a_{i_1})(a_{i_1+1} \circ \cdots \circ a_{i_1+i_2}) \cdots (a_{k-i_l+1} \circ \cdots \circ a_k).$$

For example,

$$(2, 1, 3)[z_2^2 z_1 z_3 z_1^2] = (z_2 \circ z_2)z_1(z_3 \circ z_1 \circ z_1) = z_4 z_1 z_5.$$
Formal Power Series

Now let $\mathcal{P}$ be the set of formal power series

$$f = c_1 t + c_2 t^2 + c_3 t^3 + \cdots$$

with $c_1 \neq 0$. Given $f \in \mathcal{P}$, we can define a $\mathbb{Q}$-linear function $\Psi_f : \mathcal{H}^1 \rightarrow \mathcal{H}^1$ by

$$\Psi_f(z_{a_1} \cdots z_{a_k}) = \sum_{(i_1, \ldots, i_l) \in \mathcal{C}(k)} c_{i_1} \cdots c_{i_l} [z_{a_1} \cdots z_{a_k}],$$

where $\mathcal{C}(k)$ is the set of compositions of $k$. For example, if $f = e^t - 1$ (so that $c_i = (i!)^{-1}$) then

$$\Psi_f(z_2 z_1 z_3) = z_2 z_1 z_3 + \frac{1}{2} (z_3^2 + z_2 z_4) + \frac{1}{6} z_6.$$
Now if

\[ f = c_1 t + c_2 t^2 + \cdots \quad \text{and} \quad g = d_1 t + d_2 t^2 + \cdots \]

are two elements of \( P \), they have a “functional composition”

\[
\begin{align*}
    f \circ g &= c_1(d_1 t + d_2 t^2 + \cdots) + c_2(d_1 t + d_2 t^2 + \cdots)^2 + \cdots \\
    &= c_1 d_1 t + (c_1 d_2 + c_2 d_1^2)t^2 + \cdots
\end{align*}
\]

in \( P \), and the following result holds.

**Theorem**

For \( f, g \in P \), \( \Psi_{f \circ g} = \Psi_f \Psi_g \).
The functions $\Psi_f$ need not preserve the algebra structures $(\mathcal{H}_1, \ast)$ or $(\mathcal{H}_1, \star)$, but we will shortly see some examples that do. A key result from my work with Ihara is the following.

**Theorem**

If $f = c_1 t + c_2 t^2 + \cdots \in \mathcal{P}$, then

$$\Psi_f \left( \frac{1}{1 - tz_i} \right) = \frac{1}{1 - f \circ (tz_i)}.$$  

Here $f \circ (tz_i)$ means

$$tc_1 z_i + t^2 c_2 z_i \circ z_i + t^3 c_3 z_i \circ z_i \circ z_i + \cdots$$

$$= tc_1 z_i + t^2 c_2 z_2i + t^3 c_3 z_3i + \cdots.$$
Specific Examples

Evidently $\Psi_t = \text{id}$. The function $T = \Psi_{-t}$ sends a word $w$ to $(-1)^{\text{deg} w} w$. In fact we have the following result.

**Proposition**

$$T : (\mathcal{H}^1, \star) \rightarrow (\mathcal{H}^1, \star) \text{ is an algebra homomorphism, and so is } T : (\mathcal{H}^1, \star) \rightarrow (\mathcal{H}^1, \star).$$

Clearly $T^2 = \text{id}$, so $T$ is an isomorphism. Here are two more functions: $\exp = \Psi_{e^t - 1}$, with inverse $\log = \Psi_{\log(1+t)}$. In my 2000 paper I proved the following.

**Theorem**

$\exp : (\mathcal{H}^1, \shuffle) \rightarrow (\mathcal{H}^1, \star)$ is an isomorphism of algebras, where $\shuffle$ is the usual shuffle product on $\mathcal{H}^1 = \mathbb{Q}\langle z_1, z_2, \ldots \rangle$.

This allows one to deduce the algebra structure of $(\mathcal{H}^1, \star)$ from known results about $(\mathcal{H}^1, \shuffle)$. 
But perhaps the most interesting function is $\Sigma = \Psi_t^{-1}$. Using the composition theorem above it is easy to show that $T\Sigma T = \Sigma^{-1}$ and $\Sigma = \exp T \log T$. The key property of $\Sigma$ is the following.

**Theorem**

$\Sigma : (\mathcal{H}^1, \ast) \rightarrow (\mathcal{H}^1, \ast)$ is an algebra isomorphism such that $\zeta^*(w) = \zeta(\Sigma(w))$ for all words $w$ of $\mathcal{H}^0$.

Having the isomorphism $\Sigma$ turns out to be quite useful. One can show that

$$\sum_{n \geq 0} \zeta(z^n_k) t^n = \exp \left( \sum_{i \geq 1} \frac{(-1)^{i-1} \zeta(ik)}{i} t^i \right)$$
If we call this generating function $Z_k(t)$, then the theorems above imply
\[ \sum_{n \geq 0} \zeta^*(z^n_k) t^n = \frac{1}{Z_k(-t)}. \]

Thus from the well-known result
\[ \zeta(z^n_2) = \frac{\pi^{2n}}{(2n + 1)!}, \quad \text{i.e.,} \quad Z_2(t) = \frac{\sinh \pi \sqrt{t}}{\pi \sqrt{t}}, \]

we get
\[ \sum_{n \geq 0} \zeta^*(z^n_2) t^n = \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}}, \]

and thus
\[ \zeta^*(z^n_2) = \frac{(-1)^n 2(2^{2n-1} - 1) B_{2n} \pi^{2n}}{(2n)!}. \]
Σ takes MZVs to MZSVs cont’d

But one can do much more. There is the following identity in \( \mathcal{H}^1[[t]] \).

**Theorem (Ihara-Kajikawa-Ohno-Okuda, H-Ihara)**

*For any positive integers \( i, j \),*

\[
\sum \left( \frac{1}{1 - t z_i z_j} \right) = \frac{1}{1 - t z_i z_j} \sum \left( \frac{1}{1 - t z_{i+j}} \right).
\]

Applying \( \zeta \) to the theorem with \( i = 2, j = 1 \) gives

\[
\sum_{n \geq 0} \zeta^*((z_2 z_1)^n) t^n = \sum_{p \geq 0} \zeta((z_2 z_1)^p) t^p \sum_{q \geq 0} \zeta^*(z_3^q) t^q;
\]

using the duality relation \( \zeta((z_2 z_1)^n) = \zeta(z_3^n) \), this implies
Theorem \[ \sum_{n \geq 0} \zeta^*(z_2 z_1)^n t^n = \frac{Z_3(t)}{Z_3(-t)}. \]

Then (via a lemma from the theory of symmetric functions)

\[ \zeta^*(z_2 z_1)^n = \sum_{i_1 + 3i_3 + 5i_5 + \cdots = n} 2^{i_1 + i_3 + i_5 + \cdots} \zeta(3)^{i_1} \zeta(9)^{i_3} \zeta(15)^{i_5} \cdots \frac{1^{i_1} i_1! 3^{i_3} i_3! 5^{i_5} i_5! \cdots}{n!}. \]

Similarly, from the Zagier-Broadhurst identity

\[ \zeta((z_3 z_1)^n) = \frac{2 \pi^{4n}}{(4n + 2)!}, \quad \text{i.e.,} \quad \sum_{n \geq 0} \zeta((z_3 z_1)^n) t^n = Z_4 \left( \frac{t}{4} \right) \]

we get

\[ \sum_{n \geq 0} \zeta^*((z_3 z_1)^n) t^n = \frac{Z_4 \left( \frac{t}{4} \right)}{Z_4(-t)} = \frac{\cosh(\pi \sqrt{t}) - \cos(\pi \sqrt{t})}{\sinh(\pi \sqrt{t}) \sin(\pi \sqrt{t})}. \]
Yamamoto’s Product

In fact, for any \( r \in \mathbb{Q} \) we can define \( \Sigma^r \) as \( \Psi_{\frac{t}{1-rt}} \). The composition theorem then gives us the following.

**Proposition**

\[
\text{For any } r, s \in \mathbb{Q}, \ \Sigma^r \Sigma^s = \Sigma^{r+s}.
\]

Ihara and I got some interesting results involving fractional \( \Sigma^r \) in our 2012 work, e.g.,

\[
\Sigma^r \left( \frac{1}{1 - tz_i} \right) \ast \Sigma^{1-r} \left( \frac{1}{1 + tz_i} \right) = 1
\]

for any rational \( r \), but Shuji Yamamoto did something even better in his 2013 paper; he managed to define a product \( \ast \) on \( \mathcal{H}_1 \) so that \( \Sigma^r(u \ast v) = \Sigma^r(u) \ast \Sigma^r(v) \).
Yamamoto’s Product cont’d

The definition of \( r^* \) is similar to that of \( * \), but with an interesting twist. For words \( w \) of \( \mathcal{H}_1 \), define \( w^{r*} 1 = 1^* w = w \); for all \( i, j \geq 1 \) define \( z_i^{r*} z_j = z_i z_j + z_j z_i + (1 - 2r)z_{i+j} \); and for words \( u, v \) of \( \mathcal{H}_1 \) that are not both 1 and for \( i, j \geq 1 \), define

\[
 z_i^r u^{r*} z_j^r v = z_i (u^{r*} z_j^r v) + z_j (z_i^{r*} u^{r*} v) + (1 - 2r)z_{i+j}^r (u^{r*} v) \\
+ (r^2 - r)z_{i+j}^r (u^{r*} v).
\]

The novel element is the fourth term on the right-hand side. Thus, e.g.,

\[
z_2^r z_3 z_1 = z_3 z_2 z_1 + z_2 z_1 z_3 + z_3 z_2 z_1 + (1 - 2r)(z_3^2 + z_5 z_1) + (r^2 - r)z_6.
\]

Note that \( r^* \) reduces to \( * \) if \( r = 0 \) and to \( \star \) if \( r = 1 \).
The additional term complicates computation of the product. Even in the case of powers of \( z_1 \), things get interesting. Let \( p_n^{(r)}(w) \) be the \( n \)th power of \( w \) using the \( r \)-product. For the \( \ast \)-product (i.e., \( r = 0 \)) there is the well-known result

\[
p_n^{(0)}(z_1) = \sum_{\lambda \vdash n} \binom{n}{\lambda} m_\lambda,
\]

where the sum is over partitions \( \lambda \) of \( n \), \( \binom{n}{\lambda} \) is the multinomial coefficient, and \( m_\lambda \) is the monomial symmetric function corresponding to \( \lambda \) (in variables for which \( z_1, z_2, \ldots \) are power sums). For the \( \ast \)-product (i.e., \( r = 1 \)) the formula only adds an alternation in sign:

\[
p_n^{(1)}(z_1) = \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} \binom{n}{\lambda} m_\lambda,
\]
where \( \ell(\lambda) \) is the number of parts of \( \lambda \). But for general \( r \) this becomes

\[
p_n^{(r)}(z_1) = \sum_{\lambda \vdash n} \binom{n}{\lambda}^{\ell(\lambda)} \prod_{i=1}^{\ell(\lambda)} a_{\lambda_i}(r)m_{\lambda}
\]

for

\[
a_p(r) = \sum_{j=1}^{p} \left\{ \frac{p}{j} \right\} j!(-r)^{j-1},
\]

where \( \left\{ \frac{p}{j} \right\} \) is the number of partitions of \( \{1, 2, \ldots, p\} \) with \( j \) blocks (Stirling number of the second kind).
Yamamoto proved a version of the sum theorem for his product. The sum of $\zeta^r(w) = \zeta(\Sigma^r(w))$ over all words of degree $n$ and $y$-degree $k < n$ in $\mathfrak{H}^0$ is

$$\sum_{i=0}^{k-1} \binom{n-1}{i} r^i (1 - r)^{k-1-i} \zeta(n).$$

This follows from the following version of the cyclic sum theorem for Yamamoto’s product.

**Theorem (Yamamoto)**

*For words $w$ of $\mathfrak{H}^1$ not a power of $y$,*

$$\zeta^r(\tau C \tau(w)) = (1 - r)\zeta^r(C(w)) + r^k n \zeta(n + 1),$$

*where $n = \deg w$ and $k$ is the $y$-degree of $w$.*
Note that the latter result interpolates nicely between the (Hoffman-Ohno) cyclic sum theorem for MZVs \((r = 0)\) and the (Ohno-Wakabayashi) cyclic sum theorem for MZSVs \((r = 1)\). It seems that the cyclic sum theorem is the “right” result, in the sense that it (unlike the derivation theorem) (1) implies the sum theorem and (2) generalizes nicely to \(\zeta^r\).
The product $\circ$ leads to an interesting conjecture. Let $w_1, \ldots, w_k$ be words of $\mathcal{H}^1$. For $m \leq k - 1$, define $R_{k,m}(w_1, \ldots, w_k)$ as the sum of the $\binom{k-1}{m}$ terms of the form

$$w_1 \Box w_2 \Box \cdots \Box w_k,$$

where $m$ of the boxes are $\circ$ and the others are empty. E.g.,

$$R_{3,0}(w_1, w_2, w_3) = w_1 w_2 w_3$$
$$R_{3,1}(w_1, w_2, w_3) = w_1 \circ w_2 w_3 + w_1 w_2 \circ w_3$$
$$R_{3,2}(w_1, w_2, w_3) = w_1 \circ w_2 \circ w_3.$$

Now let $\lambda$ be a partition of $n$ with $k \leq n$ parts. For $m \leq k - 1$, define
New Sum Theorems?, cont’d

\[ P_{k,m}(\lambda) = \sum_{\{|w_1|, \ldots, |w_k|\}=\lambda} \zeta(xR_{k,m}(w_1, \ldots, w_k)). \]

For example, if \( \lambda = (2, 1) \) then

\[ P_{2,1}(\lambda) = \zeta(xy^3 + x^2y^2 + xy^3 + xyxy) = 2\zeta(2, 1, 1) + \zeta(3, 1) + \zeta(2, 2). \]

Conjecture

For any partition \( \lambda \) of \( n \) with \( k \) parts,

\[ P_{k,m}(\lambda) = \frac{(k - 1)!n}{|\text{Sym}(\lambda)|} \zeta(n + 1) \]

for all \( m \leq k - 1 \), where \( \text{Sym}(\lambda) \) is the group that exchanges identical parts of \( \lambda \).
Some observations are in order. First, duality implies that

$$P_{k,m}(\lambda) = P_{k,k-1-m}(\lambda)$$

for any $$m \leq k - 1$$. Of course the conjecture implies that $$P_{k,m}(\lambda)$$ doesn’t depend on $$m$$ at all.

Second, the conjecture follows from the (classical) sum theorem if $$k = n$$ (i.e., if $$\lambda = (1,1,1,\ldots,1)$$) or if $$k = n - 1$$ (i.e., if $$\lambda = (2,1,1,\ldots,1)$$). Third, it holds in the case $$\lambda = (3,1,1,\ldots,1)$$, but here one must use the cyclic sum theorem. The same is true of numerous special cases.
Finally, the conjecture implies the following result. For positive integers $p, q$, let

$$S_{k,m}(p, q) = \sum_{i_1 + \cdots + i_k = p} \sum_{|w_s| = i_s + j_s + 1} \sum_{j_1 + \cdots + j_k = q} 1 \leq s \leq k \zeta(xR_{k,m}(w_1, \ldots, w_k)).$$

Then $S_{k,m}(p, q)$ is evidently symmetric in $p$ and $q$. The conjecture above implies that

$$S_{k,m}(p, q) = \frac{p + q + k}{k} \binom{p + k - 1}{k - 1} \binom{q + k - 1}{k - 1} \zeta(p + q + k + 1).$$
Addendum: Lemma on Symmetric Functions

Here is the lemma used to derive the equation for $\zeta^*((z_2 z_1)^n)$.

**Lemma**

If $E(t), H(t)$ are respectively the generating functions for the elementary and complete symmetric functions, then the coefficient of $t^n$ in $E(t)H(t)$ is

$$\sum_{i_1+3i_3+5i_5+\cdots=n} \frac{2^{i_1+i_3+i_5+\cdots} p_1^{i_1} p_3^{i_3} p_5^{i_5} \cdots}{i_1!^1 i_3!^3 i_5!^5 \cdots}.$$ 

This follows from two formulas that can be found in Macdonald’s book *Symmetric Functions and Hall Polynomials*:

$$E(t) = \sum_{\lambda} \epsilon_\lambda z_\lambda^{-1} p_\lambda t^{\mid \lambda \mid}, \quad H(t) = \sum_{\lambda} z_\lambda^{-1} p_\lambda t^{\mid \lambda \mid}$$
where in each case the sum is over all partitions $\lambda$, and for $\lambda = (\lambda_1, \lambda_1, \ldots)$,

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots, \quad \epsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)}, \quad z_\lambda = \prod_{i \geq 1} m_i(\lambda) i^{m_i(\lambda)}$$

($|\lambda|$ is the sum of the parts of $\lambda$ and $m_i(\lambda)$ is the number of parts of $\lambda$ equal to $i$, so $\ell(\lambda) = m_1(\lambda) + m_2(\lambda) + \cdots$ and $|\lambda| = m_1(\lambda) + 2m_2(\lambda) + \cdots$). From the formulas above,

$$E(t)H(t) = \sum_{\mu, \lambda} \frac{\epsilon_\mu}{z_\mu z_\lambda} p_{\mu \cup \lambda} t^{|\mu| + |\lambda|} = \sum_{\nu} p_\nu t^{|\nu|} \sum_{\mu \cup \lambda = \nu} \frac{\epsilon_\mu}{z_\mu z_\lambda};$$

so it suffices to show

$$\sum_{\mu \cup \lambda = \nu} \frac{\epsilon_\mu}{z_\mu z_\lambda} = \begin{cases} 2^{\ell(\nu)} z_{\nu}^{-1}, & \text{if } \nu \text{ has all parts odd;} \\ 0, & \text{otherwise.} \end{cases}$$
This is equivalent to

$$\sum_{\mu \cup \lambda = \nu} \epsilon_\mu \prod_{i \geq 1} \left( \frac{m_i(\nu)}{m_i(\mu)} \right) = \begin{cases} 2^{\ell(\nu)}, & \text{if } \nu \text{ has all parts odd;} \\ 0, & \text{otherwise.} \end{cases}$$

But the latter is immediate if we note that

$$\epsilon_\mu = (-1)^{\sum_i (i-1)m_i(\mu)}$$

and rewrite the left-hand side as

$$\sum_{\mu \cup \lambda = \nu} \prod_{i \geq 1} (-1)^{(i-1)m_i(\mu)} \left( \frac{m_i(\nu)}{m_i(\mu)} \right) = \prod_{m_i(\nu) \neq 0} \sum_{j=0}^{m_i(\nu)} (-1)^{(i-1)j} \binom{m_i(\nu)}{j}.$$