Multiple Zeta Values and Their Extended Family

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Outline

The 1990's as a golden age

Multiple zeta values (MZVs) are defined by

$$\zeta(i_1,\dots,i_k)=\sum_{n_1>\cdots>n_k\geq 1}\frac{1}{n_1^{i_1}\cdots n_k^{i_k}}$$

for positive integers $i_1,\dots,i_k$ with $i_1>1$. The sum of the indices $i_1+\cdots+i_k$ is called the weight, and k is the depth. For depth ≤ 2 the study of these numbers goes back to Euler, but the general depth case emerged as an area of active research only in the 1990's. At that time MZVs appeared simultaneously in knot theory and perturbative quantum field theory. The field suddenly mushroomed, and in retrospect the 1990's appear as a golden age of exciting developments.

The golden age cont'd

I was introduced to this subject in 1988 by my colleague Courtney Moen, who told me about the "sum conjecture": the sum of all MZVs of fixed depth and weight n is $\zeta(n)$. For example,

$$\zeta(2,1,1,1) = \zeta(3,1,1) + \zeta(2,2,1) + \zeta(2,1,2) \\ = \zeta(4,1) + \zeta(3,2) + \zeta(2,3) = \zeta(5).$$

Euler proved this for depth 2, and Courtney proved it for depth 3. I plunged in, and though I couldn't prove the sum theorem in I obtained several results, which appeared in my 1992 paper:

1. the duality formula (which I could only prove for special cases);
2. the derivation formula;
3. the symmetric sum theorem.

The golden age cont'd

While I failed to prove the sum conjecture, my derivation formula reduced Moen's 50-page proof of the depth 3 case to three pages, which we published just in time: Andrew Granville and Don Zagier independently proved the general case. Unlike the sum theorem, the duality theorem equates pairs of MZVs, e.g., $\zeta(3,2) = \zeta(2,2,1)$. It has a nice general proof, which I shall describe next. To describe it and the derivation formula it will be useful to introduce some algebraic notation. I'll come back to the symmetric sum theorem in due course.

MZVs as sums and integrals

I would have proved duality in general back in 1992 had I realized that MZVs can be represented by iterated integrals as well as by series. For example,

$$\int_0^1 \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_1}{1-t_1} = \int_0^1 \frac{dt_3}{t_3} \int_0^{t_3} \sum_{i\ge 1} \frac{t_2^i}{i} \frac{dt_2}{1-t_2} =$$

$$\int_0^1 \sum_{i,j\ge 1} \frac{t_3^{i+j}}{i(i+j)} \frac{dt_3}{t_3} = \sum_{i,j\ge 1} \frac{1}{i(i+j)^2} = \zeta(2,1).$$

Algebraic notation

Computations like the preceding can be represented using the notation

$$x \sim \frac{dt}{t}, \quad y \sim \frac{dt}{1-t},$$

so that the form integrated on the preceding slide is xy^2. Thinking of ζ as the function that evaluates the iterated integral, $\zeta(2,1) = \zeta(xy^2)$ and more generally

$$\zeta(i_1, \dots, i_k) = \zeta(x^{i_1-1}y \cdots x^{i_k-1}y).$$

The monomials "live" in the noncommutative polynomial ring $\mathbb{Q}\langle x, y\rangle$. The change of variable $t \mapsto 1-t$ corresponds to the antiautomorphism τ of $\mathbb{Q}\langle x, y\rangle$ that exchanges x and y, rso $\zeta(\tau(w)) = \zeta(w)$ for monomials w: this is the duality theorem.

Algebraic notation cont'd

Actually the iterated integral only converges if the monomial begins in x and ends in y. Let $\mathfrak{H}$ be the underlying rational vector space of $\mathbb{Q}\langle x, y\rangle$, $\mathfrak{H}^0$ the subspace generated by monomials starting with x and ending in y, together with the empty monomial 1. There is a commutative product on $\mathfrak{H}$, namely the shuffle product $\sqcup\!\sqcup$, and $(\mathfrak{H}^0, \sqcup\!\sqcup)$ is a subalgebra of $(\mathfrak{H}, \sqcup\!\sqcup)$. We have, e.g.,

$$xy \sqcup\!\sqcup\, xy = 2xyxy + 4x^2y^2. \tag{1}$$

In fact shuffle product corresponds, via integration by parts, to the product of iterated integrals, so ζ becomes a homomorphism from $(\mathfrak{H}^0, \sqcup\!\sqcup)$ to the reals.

The other product

Thus corresponding to Eq. (1) we have

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1) \tag{2}$$

But we can also multiply MZVs as series, e.g.,

$$\zeta(2)^2 = \zeta(4) + 2\zeta(2,2), \tag{3}$$

corresponding to a multiplication $*$ on $\mathfrak{H}^0$ with $xy * xy = x^3y + 2xyxy$. (This is sometimes called the "stuffle" product.) Since $\zeta(2)^2 = \frac{\pi^4}{36} = \frac{5}{2}\zeta(4)$, Eq. (3) implies that $\zeta(2,2) = \frac{3}{4}\zeta(4)$. Comparing with Eq. (2) gives $\zeta(4) = 4\zeta(3,1)$, or $\zeta(3,1) = \frac{1}{4}\zeta(4)$.

The other product cont'd

More generally, one defines $*$ inductively by

$$x^p yw * x^q yv = x^p y(w * x^q yv) + x^q y(x^p yw * v) + x^{p+q+1} y(w * v)$$

for monomials w, v in $\mathfrak{H}^0$. So now we have two commutative products on $\mathfrak{H}^0$, the shuffle $\sqcup\!\sqcup$ and the stuffle $*$, with both $\zeta : (\mathfrak{H}^0, \sqcup\!\sqcup) \to \mathbb{R}$ and $\zeta : (\mathfrak{H}^0, *) \to \mathbb{R}$ being homomorphisms. Actually the shuffle and stuffle product can be defined on the subspace $\mathfrak{H}^1$ of $\mathfrak{H}$ generated by 1 and all monomials that end in y, and in my 1997 paper (in which I adopted an algebraic framework) I recognized $(\mathfrak{H}^1, *)$ as the algebra QSym of quasi-symmetric fuctions. This has the benefit of allowing one to use symmetric fuctions, since Sym is a subalgebra of QSym.

The derivation formula

The algebraic framework also gives a succinct statement of the derivation formula. Let D be the derivation on $\mathbb{Q}\langle x, y\rangle$ such that $D(x) = 0$ and $D(y) = xy$. Then D takes $\mathfrak{H}^0$ to itself, and $\bar{D} = \tau D \tau$ is also a derivation taking $\mathfrak{H}^0$ to itself. The derivation formula is

$$\zeta(D(w)) = \zeta(\bar{D}(w))$$

for all monomials w in $\mathfrak{H}^0$. For example, since

$$D(xy^2xy) = x^2y^2xy + xyxyxy + xy^2x^2y,$$
$$\bar{D}(xy^2xy) = xy^2xy^2 + xy^3xy$$

we have

$$\zeta(3,1,2) + \zeta(2,2,2) + \zeta(2,1,3) = \zeta(2,1,2,1) + \zeta(2,1,1,1,2).$$

Generalizations and analogue

Yasuo Ohno proved a general identity that includes the duality and derivation formulas as special cases. Masanobu Kaneko and Kenataro Ihara gave another generalization of the derivation theorem: if ∂_n is the derivation of $\mathbb{Q}\langle x, y\rangle$ with

$$\partial_n(x) = -\partial_n(y) = x(x+y)^{n-1}y,$$

then $\zeta(\partial_n(w)) = 0$ for $w \in \mathfrak{H}^0$. This generalizes the derivation formula since $\partial_1 = \bar{D} - D$.

In 1999 I conjectured another result, the cyclic sum theorem. Yasuo Ohno proved it while he was here in 2000. We published a joint paper on this and related results in 2003.

Cyclic sum theorem

Here's an algebraic description of the cyclic sum theorem. Define the "cyclic derivation" C on $\mathfrak{H}$ by $C(w) = \bar{\mu}\hat{C}(w)$, where $\hat{C} : \mathfrak{H} \to \mathfrak{H} \otimes \mathfrak{H}$ is the derivation taking x to 0 and y to $y \otimes x$ and $\bar{\mu} : \mathfrak{H} \otimes \mathfrak{H} \to \mathfrak{H}$ is reversed multiplication. Then the cyclic sum theorem says that $\zeta(C(w) - \tau C \tau(w)) = 0$ for any $w \in \mathfrak{H}^1$ not a power of y. While C isn't a derivation, it behaves better than D on periodic words, e.g.,

$$C(xyxyxy) = 3x^2yxyxy$$

while

$$D(xyxyxy) = x^2yxyxy + xyx^2yxy + xyxyx^2y.$$

Also, the cyclic sum theorem implies the sum theorem mentioned earlier.

The MZV landscape

Many wonderful identities were conjectured and mostly proved during the "golden age," in particular the Zagier-Broadhurst formula

$$\zeta(\{3,1\}_n) = \frac{2\pi^{4n}}{(4n+2)!}.$$

(Here $\{s\}_n$ denotes n repetitions of the string s.) But a broader question is this: just what subalgebra $\mathcal{MZV} \subset \mathbb{R}$ is generated by MZVs (over $\mathbb{Q}$)? Are they all just rational polynomials in the zeta values $\zeta(n)$, $n \geq 2$, as in

$$\zeta(2,3,2) = \frac{75}{8}\zeta(7) - \frac{11}{2}\zeta(2)\zeta(5)?$$

It doesn't appear so. Although all MZVs of weight 7 or less are rational polynomials in the ordinary zeta values, $\zeta(6,2)$ doesn't appear to be.

The MZV landscape cont'd

Now all known relations of MZVs preserve weight, so we think of the algebra $\mathcal{MZV}$ as graded. One can then ask about the dimension of the rational vector space $\mathcal{MZV}_n$ of weight-n MZVs. Don Zagier conjectured that $\dim_{\mathbb{Q}} \mathcal{MZV}_n = P_n$, the nth Padovan number, defined by $P_1 = 0$, $P_2 = P_3 = 1$, and $P_n = P_{n-2} + P_{n-3}$ for $n \geq 4$. While putting a lower bound on $\dim_{\mathbb{Q}} \mathcal{MZV}_n$ seems out of reach, it was proved (independently) by Goncharov and Terasoma (in 2001 and 2002 respectively) that $\dim_{\mathbb{Q}} \mathcal{MZV}_n \leq P_n$.

In my 1997 paper I conjectured that $\mathcal{MZV}_n$ has basis H_n, where H_n is the set of weight-n MZVs with only arguments 2 or 3, e.g., $H_7 = \{\zeta(3,2,2), \zeta(2,3,2), \zeta(2,2,3)\}$; H_n is easily seen to have cardinality P_n. In 2011 Francis Brown proved (with the help of a supporting result from Don Zagier) that H_n spans $\mathcal{MZV}_n$.

The MZV landscape cont'd

We call an MZV reducible if it is a rational polynomial in the ordinary zeta values $\zeta(i)$. Thus $\zeta(6,2)$ is an example of an (apparently) non-reducible MZV of minimal weight. In any case one can't form $P_8 = 4$ independent quantities by taking products of zeta values of lesser weight; all weight-8 products of zeta values are rational linear combinations of $\{\zeta(8), \zeta(2)\zeta(3)^2, \zeta(3)\zeta(5)\}$. As it turns out, all MZVs of weight 9 are reducible, but in weights 10 and higher this doesn't seem to be the case.

One can do symbolic calculation using known MZV relations to express all MZVs of weight n as $\mathbb{Q}$-linear combinations of P_n quantities. This was done by the Lille group (Bigotte et. al.) through weight 14 around 1999. More recently tables through weight 24 were obtained by physicists Johannes Blümlein et. al. and put online as the "Multiple Zeta Value Data Mine".

The MZV landscape cont'd

One can also ask where all the MZV relations come from. We've seen above that relations can be obtained by comparing the shuffle and stuffle products. Conjecturally (Ihara, Kaneko and Zagier 2016) all such relations arise this way, provided we consider the products on $\mathfrak{H}^1$ and not just on $\mathfrak{H}^0$. For example,

$$y \sqcup\!\sqcup w - y * w = \bar{D}(w) - D(w)$$

for $w \in \mathfrak{H}^0$.

Quasi-shuffle products

In 2001 I introduced "quasi-shuffle products" to extend the algebraic approach. The basic idea is as follows. Suppose A is a countable set and $\diamond$ is a commutative product on $\mathbb{Q}A$ so that, for $a, b \in A$, $a \diamond b \in \mathbb{Q}A$ is a finite sum. Now define a new product $*$ on the noncommutative polynomial algebra $\mathbb{Q}\langle A \rangle$ inductively by setting $1 * w = w * 1 = w$ for all monomials w and

$$au * bv = a(u * bv) + b(au * v) + (a \diamond b)(u * v)$$

for $a, b \in A$ and monomials $u, v \in \mathbb{Q}A$. Then $(\mathbb{Q}\langle A \rangle, *)$ is commutative and associative, and in fact a polynomial algebra.

A motivating example

If $A = \{z_1, z_2, \dots\}$ and $z_i \diamond z_j = z_{i+j}$, then $(\mathbb{Q}\langle A\rangle, *)$ is what we called $(\mathfrak{H}^1, *)$ above. As already mentioned, $(\mathfrak{H}^1, *)$ is isomorphic to the algebra QSym of quasi-symmetric functions. One can deduce from the quasi-shuffle product construction that $(\mathfrak{H}^1, *)$ is a polynomial algebra without invoking the result for quasi-symmetric functions. If $\mathbb{Q}\langle A\rangle^0$ is the subalgebra generated by 1 and monomials that don't start with z_1, then there is a homomorphism $\zeta : \mathbb{Q}\langle A\rangle^0 \to \mathbb{R}$ sending $z_{i_1} \cdots z_{i_k}$ to $\zeta(i_1, \dots, i_k)$.

More algebra

The algebraic construction of $(\mathbb{Q}\langle A\rangle, *)$ goes further. If we define $\Delta : \mathbb{Q}\langle A\rangle \to \mathbb{Q}\langle A\rangle \otimes \mathbb{Q}\langle A\rangle$ by deconcatenation, i.e.,

$$\Delta(a_1a_2\cdots a_n) = a_1a_2\cdots a_n \otimes 1 + a_1a_2\cdots a_{n-1} \otimes a_n + \cdots + 1 \otimes a_1a_2\cdots a_n,$$

for $a_1,\dots,a_n \in A$, then $(\mathbb{Q}\langle A\rangle, *, \Delta)$ is a connected filtered bialgebra, and hence a Hopf algebra. In fact, in my 1997 paper I gave a formula for the antipode.

More algebra cont'd

Since we'll need it shortly, I'll give it here. First, consider some linear functions on $\mathbb{Q}\langle A\rangle$: the function R that reverses monomials, the function T given by $T(w) = (-1)^{\ell(w)}w$, where $\ell(w)$ is the length of w, and the function Σ that adds up all contractions of a monomial, e.g.,

$$\Sigma(a_1 a_2 a_3) = a_1 a_2 a_3 + a_1 \diamond a_2 a_3 + a_1 a_2 \diamond a_3 + a_1 \diamond a_2 \diamond a_3.$$

Then R and T are involutions that commute with each other, while Σ has infinite order: R commutes with Σ, but $T\Sigma = \Sigma^{-1}T$. Our formula for the antipode of $(\mathbb{Q}\langle A\rangle, *, \Delta)$ is $S = \Sigma T R$.

Alternating MZVs

Alternating MZVs are like MZVs with powers of -1 in the numerator, e.g.,

$$\zeta(\bar{2},1,\bar{3}) = \sum_{n_1>n_2>n_3\geq 1} \frac{(-1)^{n_1+n_3}}{n_1^2 n_2 n_3^3}.$$

Convergence requires that the first entry in the argument is not an unbarred 1. Note that

$$\zeta(\bar{1}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\log 2.$$

Alternating MZVs have been around as long as MZVs have; in fact, a physics paper of David Broadhurst from 1986 features $\zeta(\bar{6},\bar{2})$.

Alternating MZVs cont'd

There are lots more alternating MZVs than MZVs; indeed, if we denote the algebra generated by alternating MZVs by $\mathcal{AMZV}$, then (as with MZVs) $\mathcal{AMZV}$ appears to be graded by weight, and it is a long-standing conjecture that $\dim_{\mathbb{Q}} \mathcal{AMZV}_n = F_{n+1}$, the $(n+1)$st Fibonacci number. A number of interesting identities involving alternating MZVs are known, for example

$$\zeta(\{\bar{2},1\}_n) = \frac{1}{8^n}\zeta(\{3\}_n),$$

which was proved by Jianqiang Zhao in 2016, nineteen years after it was conjectured.

Alternating MZVs cont'd

The quasi-shuffle product construction just outlined easily accomodates alternating MZVs. Let

$$A = \{z_{n,\epsilon} \mid n \in \mathbb{Z}^+, \ \epsilon \in \mathbb{Z}/2\mathbb{Z}\}$$

with product $z_{n,\epsilon} \diamond z_{m,\eta} = z_{n+m,\epsilon+\eta}$, the addition in the second subscript being understood mod 2. The quasi-shuffle algebra $(\mathbb{Q}\langle A\rangle, *)$ is "larger" than QSym, but is still a polynomial algebra. If $\mathbb{Q}\langle A\rangle^0$ is the subalgebra of $\mathbb{Q}\langle A\rangle$ generated by 1 and monomials that don't begin with $z_{1,0}$, then there is a homomorphism from $\mathbb{Q}\langle A\rangle^0$ to $\mathbb{R}$ sending, e.g., $z_{1,1}z_{2,0}z_{3,1}$ to $\zeta(\bar{1}, 1, \bar{3})$.

Multiple zeta-star values

Now I want to return to $\mathcal{MZV}$, and talk about multiple zeta-star values (MZSVs), which can be defined by

$$\zeta^\star(i_1, \dots, i_k) = \sum_{n_1 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}.$$

These differ from MZVs only in having non-strict inequalities in the summation indices. Thus, e.g.,

$$\zeta^\star(3,1,2) = \zeta(3,1,2) + \zeta(4,2) + \zeta(3,3) + \zeta(6).$$

and conversely

$$\zeta(3,1,2) = \zeta^\star(3,1,2) - \zeta^\star(4,2) - \zeta^\star(3,3) + \zeta^\star(6).$$

Multiple zeta-star values cont'd

Since MZVs and MZSVs are each just sums of the other, the space of MZSVs is just $\mathcal{MZV}$; we're really just considering two different bases for the same set of numbers. In fact, in my 1992 paper I treated them on a more or less equal footing, writing $A(i_1, \dots, i_k)$ for $\zeta(i_1, \dots, i_k)$ and $S(i_1, \dots, i_k)$ for $\zeta^\star(i_1, \dots, i_k)$, but this notation didn't catch on. The sum theorem for MZVs is equivalent to the statement that

$$\sum_{i_1+\cdots+i_k=n, i_1>1} \zeta^\star(i_1, \dots, i_k) = \binom{n-1}{k-1}\zeta(n)$$

for all $n > k \geq 1$, and I showed that the symmetric sum theorem has an analogue for MZSVs.

Symmetric sum theorem

So let me describe the symmetric sum theorem. In my 1992 paper I showed that the sum of $\zeta(i_1, \dots, i_k)$ over all permutations of $i_1, \dots, i_k$ (necessarily each $i_j \geq 2$) always gives a rational polynomial in ordinary zeta values, e.g.,

$$\zeta(i_1, i_2) + \zeta(i_2, i_1) = \zeta(i_1)\zeta(i_2) - \zeta(i_1 + i_2)$$

and

$$\begin{aligned} &\zeta(i_1, i_2, i_3) + \zeta(i_1, i_3, i_2) + \zeta(i_2, i_1, i_3) + \zeta(i_2, i_3, i_1) + \\ &\zeta(i_3, i_1, i_2) + \zeta(i_3, i_2, i_1) = \zeta(i_1)\zeta(i_2)\zeta(i_3) - \zeta(i_1)\zeta(i_2 + i_3) \\ &\quad - \zeta(i_2)\zeta(i_1 + i_3) - \zeta(i_3)\zeta(i_1 + i_2) + 2\zeta(i_1 + i_2 + i_3). \end{aligned}$$

Symmetric sum theorem cont'd

The general result is as follows. Let S_k be the symmetric group on $\{1,\dots,k\}$, Π_k the set of partitions of $\{1,\dots,k\}$. For $B=\{B_1,\dots,B_l\}\in\Pi_k$, define

$$c(B)=(-1)^{k-l}(\operatorname{card} B_1-1)!\cdots(\operatorname{card} B_l-1)!$$

Theorem (Symmetric sums of MZVs)

For integers $i_1,\dots,i_k\ge 2$,

$$\sum_{\sigma\in S_k}\zeta(i_{\sigma(1)},\dots,i_{\sigma(k)})=\sum_{B=\{B_1,\dots,B_l\}\in\Pi_k} c(B)\zeta(\sum_{j\in B_1} i_j)\cdots\zeta(\sum_{j\in B_l} i_j).$$

Symmetric sums of MZSVs

Not much change is required if ζ is replaced by $\zeta^\star$ (though the latter notation didn't exist in 1992). In my 1992 paper I also proved the following.

Theorem (Symmetric sums of MZSVs)

For integers $i_1, \dots, i_k \geq 2$,

$$\sum_{\sigma \in S_k} \zeta^\star(i_{\sigma(1)}, \dots, i_{\sigma(k)}) = \sum_{B=\{B_1,\dots,B_l\} \in \Pi_k} \bar{c}(B) \zeta\Big(\sum_{j \in B_1} i_j\Big) \cdots \zeta\Big(\sum_{j \in B_l} i_j\Big),$$

where for $B = \{B_1, \dots, B_l\} \in \Pi_k$,

$$\bar{c}(B) = (\operatorname{card} B_1 - 1)! \cdots (\operatorname{card} B_l - 1)!$$

Repeated arguments

By taking $i_1 = i_2 = \cdots = i_k$ in the preceding results we can obtain formulas for repeated arguments: let $\{i\}_k$ denote k repetitions of i. Then for $i \geq 2$,

$$\zeta(\{i\}_k) = \sum_{\lambda \vdash k} \frac{(-1)^{k-\ell(\lambda)}}{m_1(\lambda)! 1^{m_1(\lambda)} m_2(\lambda)! 2^{m_2(\lambda)} \cdots} \prod_{j=1}^{\ell(\lambda)} \zeta(i\lambda_j),$$

where $\lambda \vdash k$ means $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is an integer partition of k, and $m_q(\lambda)$ is the multiplicity of q in λ. Similarly

$$\zeta^\star(\{i\}_k) = \sum_{\lambda \vdash k} \frac{1}{m_1(\lambda)! 1^{m_1(\lambda)} m_2(\lambda)! 2^{m_2(\lambda)} \cdots} \prod_{j=1}^{\ell(\lambda)} \zeta(i\lambda_j).$$

Repeated arguments cont'd

One can prove formulas for repeated *even* arguments like

$$\zeta(\{2\}_n) = \frac{\pi^{2n}}{(2n+1)!}$$

using generating functions, but these formulas work for both even and odd arguments, e.g.,

$$\zeta(3,3,3) = \frac{1}{3}\zeta(9) - \frac{1}{2}\zeta(3)\zeta(6) + \frac{1}{6}\zeta(3)^3$$

and

$$\zeta^\star(3,3,3) = \frac{1}{3}\zeta(9) + \frac{1}{2}\zeta(3)\zeta(6) + \frac{1}{6}\zeta(3)^3.$$

Neglect

But while there is a nice parallelism between sum theorems and symmetric sums theorems for MZVs and MZSVs, the duality and derivation theorems don't have nice analogues for MZSVs. After my 1992 paper I neglected them. This was a mistake, since in some ways the properties of MZSVs are nicer than those of MZVs. Yasuo Ohno and Noriko Wakabayashi brought this to my attention by proving a cyclic sum theorem for MZSVs in 2006 that's simpler the the MZV version: namely for any $w \in \mathfrak{H}^1_{n-1}$ not a power of y,

$$\zeta^\star(\tau C \tau(w)) = (n+1)\zeta(n).$$

MZSVs and quasi-shuffle products

MZSVs fit nicely into the quasi-shuffle product framework. Recall that for $A = \{z_1, z_2, \dots\}$ and $z_i \diamond z_j = z_{i+j}$, the MZVs are images of a homomorphism from the quasi-shuffle algebra $(\mathbb{Q}\langle A\rangle^0, *)$ to $\mathbb{R}$. We can define another product $\star$ on $\mathbb{Q}\langle A\rangle^0$ inductively by setting $w \star 1 = 1 \star w = w$ and

$$z_i u \star z_j v = z_i(u \star z_j v) + z_j(z_i u \star v) - z_{i+j}(u \star v).$$

Then there is a homomorphism from $(\mathbb{Q}\langle A\rangle^0, \star)$ to $\mathbb{R}$ sending $z_{i_1} \cdots z_{i_k}$ to $\zeta^\star(i_1, \dots, i_j)$. Now the two symmetric sum theorems can actually be pulled back to $\mathbb{Q}\langle A\rangle$, i.e.,

$$\sum_{\sigma \in S_k} z_{i_{\sigma(1)}} \cdots z_{i_{\sigma(k)}} = \sum_{B=\{B_1,\dots,B_l\}\in\Pi_k} c(B) z_{[B_1]} * \cdots * z_{[B_l]}$$

MZSVs and quasi-shuffle products cont'd

and

$$\sum_{\sigma\in S_k} z_{i_{\sigma(1)}}\cdots z_{i_{\sigma(k)}} = \sum_{B=\{B_1,\dots,B_l\}\in\Pi_k} \bar{c}(B) z_{[B_1]}\star\cdots\star z_{[B_l]},$$

where $[B_q]$ means $\sum_{j\in B_q} i_j$; then the theorems above follow by applying the homomorphisms ζ and $\zeta^\star$ respectively.
This viewpoint is also useful for proving a result about MZVs and MZSVs I noticed last year. We need one additional ingredient: $\mathfrak{H}^1 = \mathbb{Q}\langle A\rangle$ turns out to be $\mathfrak{H}^0[z_1] = \mathbb{Q}\langle A\rangle^0[z_1]$, so we can extend $\zeta : \mathfrak{H}^0 \to \mathbb{R}$ to $\zeta : \mathfrak{H}^1 \to \mathbb{R}[x]$ by sending z_1 to x. This sends $z_{i_1}\cdots z_{i_k}$ to $\zeta(i_1,\dots,i_k)\in\mathbb{R}$ if $i_1>1$ and to the "regularized" MZV $\zeta(i_1,\dots,i_k)\in\mathbb{R}[x]$ otherwise; e.g., $\zeta(1,3) = -\frac{5}{4}\zeta(4) + x\zeta(3)$.

A surprising pattern

Last spring I compiled some tables of MZSVs, using existing ones for MZVs, through weight 9. (As mentioned earlier, all MZVs of weight 9 are reducible, and one can use the basis $\{\zeta(9), \zeta(2)\zeta(7), \zeta(3)\zeta(6), \zeta(4)\zeta(5), \zeta(3)^3\}$ of cardinality $P_9 = 5$.) I noticed a certain pattern. For example, the coefficients of $\zeta(9)$ in various MZVs and MZSVs are as follows:

quantity	coeff. $\zeta(9)$	quantity	coeff. $\zeta(9)$
$\zeta(4,2,3)$	-59	$\zeta^\star(4,2,3)$	46
$\zeta(3,2,4)$	46	$\zeta^\star(3,2,4)$	-59
$\zeta(4,1,2,2)$	$-\frac{1187}{36}$	$\zeta^\star(4,1,2,2)$	$-\frac{839}{72}$
$\zeta(2,2,1,4)$	$\frac{839}{72}$	$\zeta^\star(2,2,1,4)$	$\frac{1187}{36}$

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$\zeta(3,2,4)$	46	$\zeta^\star(3,2,4)$	-59
$\zeta(4,1,2,2)$	$-\frac{1187}{36}$	$\zeta^\star(4,1,2,2)$	$-\frac{839}{72}$
$\zeta(2,2,1,4)$	$\frac{839}{72}$	$\zeta^\star(2,2,1,4)$	$\frac{1187}{36}$

The general result seems to be that, for compositions I,

$$\zeta^\star(I) = (-1)^{\ell(I)-1}\zeta(\bar{I}) \text{ mod decomposables,} \tag{4}$$

where $\ell(I)$ is the length of I and $\bar{I}$ is the reverse of I.

A pattern emerges, cont'd

A further comment might be useful here. Note that for n even, $\zeta(n)$ is decomposable if $n > 2$. For weight 8, we can use the basis $\{\zeta(8), \zeta(3)\zeta(5), \zeta(2)\zeta(3)^2, \zeta(6,2)\}$. Here the apparent indecomposable is $\zeta(6,2)$, and indeed

$$\zeta(5,1,2) = -\frac{145}{72}\zeta(8) + \frac{9}{2}\zeta(3)\zeta(5) - \frac{3}{2}\zeta(2)\zeta(3)^2 - \zeta(6,2)$$
$$\zeta(2,1,5) = -\frac{181}{18}\zeta(8) + \frac{21}{2}\zeta(3)\zeta(5) - \zeta(2)\zeta(3)^2 - \frac{5}{2}\zeta(6,2)$$
$$\zeta^\star(5,1,2) = -\frac{257}{36}\zeta(8) + \frac{19}{2}\zeta(3)\zeta(5) - \frac{3}{2}\zeta(2)\zeta(3)^2 - \frac{5}{2}\zeta(6,2)$$
$$\zeta^\star(2,1,5) = -\frac{235}{72}\zeta(8) + \frac{13}{2}\zeta(3)\zeta(5) - \zeta(2)\zeta(3)^2 - \zeta(6,2).$$

Proof of the identity

Above I gave the formula $S = \Sigma TR$ for the antipode of the Hopf algebra $(\mathfrak{H}^1, *, \Delta)$. On the other hand, if for a composition $I = (i_1, \dots, i_k)$ we write z_I for $z_{i_1} \cdots z_{i_k}$, then it is easy to prove by induction that

$$S(z_I) = \sum_{I_1 \sqcup I_2 \sqcup \cdots \sqcup I_p = I} (-1)^p z_{I_1} * z_{I_2} * \cdots * z_{I_p} \tag{5}$$

where $\sqcup$ is juxtaposition, e.g., $(1,2) \sqcup (3,1,2) = (1,2,3,1,2)$. Since $(\mathfrak{H}^1, *, \Delta)$ is commutative, it follows that S is an algebra homomorphism and S^2 is the identity. Applying S to both sides of Eq. (5) gives

$$z_I = \sum_{I_1 \sqcup \cdots \sqcup I_p = I} (-1)^p S(z_{I_1}) * S(z_{I_2}) * \cdots * S(z_{I_p})$$

Proof of the identity cont'd

or, since $S = \Sigma TR$,

$$z_I = \sum_{I_1 \sqcup \cdots \sqcup I_p = I} (-1)^{\ell(I)-p} \Sigma(z_{\bar{I}_1}) * \cdots * \Sigma(z_{\bar{I}_p}).$$

Now take ζ of both sides, noting that $\zeta^\star = \zeta\Sigma$, to get

$$\zeta(I) = \sum_{I_1 \sqcup \cdots \sqcup I_p = I} (-1)^{\ell(I)-p} \zeta^\star(\bar{I}_1) \cdots \zeta^\star(\bar{I}_p).$$

This implies

$$\zeta(I) = (-1)^{\ell(I)-1} \zeta^\star(\bar{I}) \text{ mod decomposables},$$

and thus Eq. (4) is proved.

Did I pull a fast one?

Note that this proof actually gives us exact formulas relating ζ and $\zeta^\star$. For example, $\zeta(4,2,3)$ is

$$\zeta^\star(3,2,4)-\zeta^\star(3)\zeta^\star(2,4)-\zeta^\star(3,2)\zeta^\star(4)+\zeta^\star(3)\zeta^\star(2)\zeta^\star(4)$$
$$=\zeta^\star(3,2,4)-\zeta(3)\zeta^\star(2,4)-\zeta^\star(3,2)\zeta(4)+\zeta(3)\zeta(2)\zeta(4)$$

But what about $\zeta(3,1,2)$? The formula says it's equal to

$$\zeta^\star(2,1,3)-\zeta^\star(2)\zeta^\star(1,3)-\zeta^\star(2,1)\zeta^\star(3)+\zeta^\star(2)\zeta^\star(1)\zeta^\star(3),$$

but this is where the "regularized" MZVs mentioned earlier come in; all occurrences of x cancel, leaving

$$\zeta(3,1,2)=\zeta^\star(2,1,3)+\frac{5}{4}\zeta(2)\zeta(4)-2\zeta(3)^2.$$

Interpolated MZVs

Interpolated MZVs are a generalization of both MZVs and MZSVs defined by Shuji Yamamoto. It's easiest to give an example:

$$\zeta^r(2,1,3) = \zeta(2,1,3) + r\zeta(3,3) + r\zeta(2,4) + r^2\zeta(4).$$

Then $r = 0$ gives MZVs, and $r = 1$ gives MZSVs. Yamamoto was able to give various results for interpolated MZVs that specialized to known results for MZVs and MZSVs, but to me the main interest of his work was that he gave a product preserved by interpolated MZVs. That is, there is a product $\overset{r}{*}$ so that $\zeta^r(v \overset{r}{*} w) = \zeta^r(v)\zeta^r(w)$.

Yamamoto's product

Now Yamamoto's product isn't a quasi-shuffle product, but it's a close relative. It can be defined as follows: let $A = \{z_1, z_2, \dots\}$ with $z_i \diamond z_j = z_{i+j}$. Then $w \overset{r}{*} 1 = 1 \overset{r}{*} w$ for all monomials w, $z_i \overset{r}{*} z_j = z_i z_j + z_j z_i + (1-2r) z_{i+j}$, and

$$z_i u \overset{r}{*} z_j v = z_i (u \overset{r}{*} z_j v) + z_j (z_i u \overset{r}{*} v) + (1-2r) z_{i+j} (u \overset{r}{*} v) \\ + (r^2 - r) z_{i+j} \diamond (u \overset{r}{*} v)$$

for monomials u, v with $uv \neq 1$. Note how this product reduces to our original quasi-shuffle product for $r = 0$ and the negated version when $r = 1$. In both those cases the fourth term is zero.

Yamamoto's product generalized

In fact, this whole definition can be carried over to the quasi-shuffle Hopf algebra: what one gets is a Hopf algebra $(\mathbb{Q}\langle A\rangle, \overset{r}{*}, \Delta)$ with antipode $\Sigma^{1-2r}TR$, where the definition of Σ is generalized to a definition of Σ^q by inserting appropriate powers of q, as in

$$\Sigma^q(a_1a_2a_3) = a_1a_2a_3 + qa_1 \diamond a_2a_3 + a_1a_2 \diamond a_3 + q^2 a_1 \diamond a_2 \diamond a_3.$$

This enables a generalization of the symmetric sum theorem: one has

$$\sum_{\sigma\in S_k} z_{i_{\sigma(1)}} \cdots z_{i_{\sigma(k)}} = \sum_{B=\{B_1,\dots,B_l\}\in\Pi_k} c_r(B) z_{[B_1]} \overset{r}{*} \cdots \overset{r}{*} z_{[B_l]},$$

where for $B = (B_1, \dots, B_l)$, $p_a(r) = (1-r)^a - (-r)^a$ and

$$c_r(B) = (-1)^{k-1} \prod_{i=1}^{l} (\operatorname{card} P_i - 1)! p_{\operatorname{card} P_i}(r).$$

Yamamoto's product generalized

So by applying ζ^r to both sides we have

$$\sum_{\sigma \in S_k} \zeta^r(i_{\sigma(1)}, \dots, i_{\sigma(k)}) = \sum_{B=\{B_1,\dots,B_l\} \in \Pi_k} c_r(B)\zeta([B_1]) \cdots \zeta([B_l]).$$

This reduces to the previous symmetric sum theorems for $r = 0, 1$; the case $r = \frac{1}{2}$ is also interesting, but we don't have time to discuss it.

We can also generalize our earlier "mod incomposables" result to get

$$\zeta^r(I) = (-1)^{\ell(I)-1}\zeta^{1-r}(\bar{I}) \text{ mod decompsables} \tag{6}$$

for compositions I. Eq. (6) can also be derived from a result of Henrik Bachmann.

Multiple t-values

Multiple t-values are "odd relatives" of MZVs. I just published a paper about them last fall. For a sequence $(i_1, \ldots, i_k)$ of positive integers with with $i_1 > 1$, let

$$t(i_1, i_2, \ldots, i_k) = \sum_{\substack{n_1 > n_2 > \cdots > n_k \geq 1 \\ n_i \, odd}} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

The algebraic theory developed earlier for MZVs applies here, so $t(i_1, \ldots, i_k)$ is the image of a monomial under a homomorphism $t : (\mathfrak{H}^0, *) \to \mathbb{R}$. The symmetric sum theorem and results on repeated arguments carry over to multiple t-values, keeping in mind that $t(i) = (1 - 2^{-i})\zeta(i)$. But the sum and duality theorems don't.

Relation to alternating MZVs

Multiple t-values can be written as sums of alternating MZVs. The most "natural" expression involves antisymmetrizing over all occurences of bars in the exponent string, e.g.,

$$t(2,3,1) = \frac{1}{8}[\zeta(2,3,1) - \zeta(\bar{2},3,1) - \zeta(2,\bar{3},1) - \zeta(2,3,\bar{1}) + \zeta(\bar{2},\bar{3},1) + \zeta(\bar{2},3,\bar{1}) + \zeta(2,\bar{3},\bar{1}) - \zeta(\bar{3},\bar{3},\bar{1})],$$

but by being clever one can halve the number of terms:

$$t(2,3,1) = \left(\frac{1}{4} - \frac{1}{64}\right)\zeta(2,3,1) + \frac{1}{4}[\zeta(\bar{2},\bar{3},1) + \zeta(\bar{2},3,\bar{1}) + \zeta(2,\bar{3},\bar{1})$$

Using tables from the Multiple Zeta Value Data Mine, I made tables of $t(I)$ through weight 7.

Trailing 1's

An interesting feature of multiple t-value relations concerns exponent strings with trailing 1's. For example,

$$t(3,2,1) = \frac{37}{112}t(6) - \frac{33}{98}t(3)^2 + \frac{1}{4}\zeta(\bar{5},1) - \frac{1}{2}t(5)\log 2 + \frac{3}{7}t(2)\zeta(3)\log 2$$

while

$$t(3,2) = -\frac{1}{2}t(5) + \frac{3}{7}t(2)\zeta(3).$$

The general principle is: trimming a trailing 1 corresponds to formal differentiation with respect to $\log 2$.

Fibonacci-number and Saha conjectures

Evidently $\mathcal{MTV}$ is a subspace of $\mathcal{AMZV}$, but how big is it? Based on my tables through weight 7, I made the following conjecture.

Conjecture

If $n \geq 2$, then $\dim_{\mathbb{Q}} \mathcal{MTV}_n$ is the nth Fibonacci number F_n.

Analogous to my 1997 basis conjecture for $\mathcal{MZV}$, Biswajyoti Saha made this more concrete by conjecturing a specific basis for $\mathcal{MTV}_n$.

Conjecture (B. Saha)

For $n \geq 2$, $\mathcal{MTV}_n$ has basis

$$C_n = \{t(a_1+1, a_2, \dots, a_r) \mid a_1 + \cdots + a_r = n-1,\ a_i \in \{1,2\}\}.$$

Conjectures cont'd

It is easy to show that card $C_n = F_n$. Also, from my tables C_n spans $\mathcal{MTV}_n$ at least for $n \le 7$.

One can also ask how $\mathcal{MZV}$ and $\mathcal{MTV}$ are related. Masanobu Kaneko and Hirofumi Tsumura make the following conjecture.

Conjecture (M. Kaneko and H. Tsumura)

A basis for $\mathcal{MZV}_n$ is given by

$$\{t(2)^k t(n_1, \ldots, n_k) \mid n_i \text{ odd} \ge 3 \text{ and } n_1 + \cdots + n_k = n - 2k\}.$$

Of course this latter conjecture implies that $\mathcal{MZV} \subset \mathcal{MTV}$. A recent result of T. Murakami is that $t(i_1, \ldots, i_k) \in \mathcal{MZV}$ if $i_1, \ldots, i_k \ge 2$.

A mod indecomposables result

We can define interpolated multiple t-values just as we defined interpolated multiple zeta values. Since (regularized) interpolated multiple t-values $t^r(I)$ are images under a homomorphism of the Hopf algebra $(\mathfrak{H}^1, *, \Delta)$, we have the result

$$t^r(I) = (-1)^{\ell(I)-1} t^{1-r}(\bar{I}) \text{ mod indecomposables.}$$

Checking my tables of multiple t-values through weight 7, this held true.

A new pattern

But I noticed something else: even more coefficients of indecomposables seemed to agree. For example,

quantity	coeff. $t(7)$	quantity	coeff. $t(7)$
$t(3,2,2)$	$\frac{3}{16}$	$t^\star(3,2,2)$	$\frac{3}{16}$
$t(2,2,3)$	$\frac{3}{16}$	$t^\star(2,2,3)$	$\frac{3}{16}$
$t(2,2,1,2)$	$-\frac{15}{32}$	$t^\star(2,2,1,2)$	$\frac{15}{32}$
$t(2,1,2,2)$	$-\frac{15}{32}$	$t^\star(2,1,2,2)$	$\frac{15}{32}$

Note there isn't even a sign change between $t(I)$ and $t(\bar{I})$. This turns out to be because all these examples are of weight 7.

A new pattern cont'd

Here are some further examples, taken from weight 8 (where $\zeta(\bar{7},1)$ is an apparent indecomposable).

quantity	coeff. $\zeta(\bar{7},1)$	quantity	coeff. $\zeta(\bar{7},1)$
$t(3,3,2)$	$\frac{935}{1368}$	$t^\star(3,3,2)$	$-\frac{935}{1368}$
$t(2,3,3)$	$-\frac{935}{1368}$	$t^\star(2,3,3)$	$\frac{935}{1368}$

A new pattern cont'd

Here are some further examples, taken from weight 8 (where $\zeta(\bar{7},1)$ is an apparent indecomposable).

quantity	coeff. $\zeta(\bar{7},1)$	quantity	coeff. $\zeta(\bar{7},1)$
$t(3,3,2)$	$\frac{935}{1368}$	$t^\star(3,3,2)$	$-\frac{935}{1368}$
$t(2,3,3)$	$-\frac{935}{1368}$	$t^\star(2,3,3)$	$\frac{935}{1368}$

Indeed the following seems to be true.

Conjecture

$$t^r(I) = (-1)^{n-1} t^r(\bar{I}) \text{ mod decomposables}$$

for any composition I of $n \geq 3$.

Note the interpolation index r is the same on both sides, and the sign depends on the weight.

A parity result for multiple t-half values

It's natural to combine this conjecture with the earlier result: we have

$$t^r(I) = (-1)^{|I|-1} t^r(\bar{I}) = (-1)^{|I|-\ell(I)} t^{1-r}(I) \text{ mod decomposables.}$$

In particular, taking $r = \frac{1}{2}$ we have

$$t^{\frac{1}{2}}(I) = (-1)^{|I|-\ell(I)} t^{\frac{1}{2}}(I) \text{ mod decomposables.}$$

This equation doesn't say much if $|I| = \ell(I) \mod 2$, but otherwise we have the following.

Proposition

If $|I| \geq 3$ and $|I| - \ell(I)$ is odd, then the conjecture implies $t^{\frac{1}{2}}(I)$ is decomposable.

A parity result for multiple t-half values cont'd

To put this in context, we recall the "parity theorem" for MZVs that's been known for some time: if I is a composition whose length and weight have opposite parity, then $\zeta(I)$ is decomposable. No such result is known for multiple t-values, and indeed $t(2,1,1)$ appears to be indecomposable. But if the conjecture above is true, then $t^{\frac{1}{2}}$ has this same parity property. I had already noticed that many multiple t-half values are much simpler than the corresponding multiple t-values; for instance,

$$t^{\frac{1}{2}}(2,1,1,1,1) = \frac{1}{16}\zeta(6) + \frac{1}{14}t(2)t(3)\log 2 + \frac{1}{8}t(4)\log^2 2 + \frac{1}{24}t(2)\log^4 2,$$

while the formula for $t(2,1,1,1,1)$ has eleven terms.

A parity result for multiple t-half values cont'd

The parity result would explain this as follows. First note that if $I = (2, 1, \ldots, 1)$, then $|I|$ and $\ell(I)$ always differ in parity. Thus any formula for

$$t^{\frac{1}{2}}(2, \underbrace{1, \ldots, 1}_{n}) \tag{7}$$

can't contain indecomposables (unless $n = 0$). But also, by our earlier statement that trimming 1's corresponds to formal differentiation by $\log 2$, any terms in a formula for (7) of form $u \log^k 2$ can't have u indecomposable for $k < n$. This rules out most of the terms appearing the formula for $t(2, 1, 1, 1, 1)$.

Multiple t- and $t^\star$-values

The combination of the theorem and conjecture above says that

$$t^\star(I) = (-1)^{|I|-\ell(I)} t(I) \text{ mod decomposables}$$

for any composition I with $|I| \geq 3$. This can be seen to be true through weight 7 from tables I've compiled, e.g.,

$$\begin{aligned} t(2,2,2,1) &= -\frac{1}{32}t(7) - \frac{3}{56}t(3)t(4) + \frac{15}{248}t(2)t(5) \\ &\quad + \frac{1}{48}t(2)\log 2 \\ t^\star(2,2,2,1) &= \frac{1}{32}t(7) + \frac{15}{56}t(3)t(4) + \frac{15}{248}t(2)t(5) \\ &\quad + \frac{61}{48}t(2)\log 2. \end{aligned}$$

A new golden age?

I don't have a method for proving the conjecture at present. Working with the Hopf algebra $(\mathfrak{H}^1, \overset{r}{*}, \Delta)$ isn't going to work, since the conjecture isn't true for MZVs. My hope is that there is some novel algebraic structure on the set $\mathcal{MTV}$ of multiple t-values that would produce a quick proof. Finding and exploring such a structure would be exciting, perhaps even the start of a new golden age!