COUNTING ORDERED COMBINATIONS

In counting combinations, sometimes the order matters. There is a difference between putting on your sock and then your shoe, versus putting on your shoe and then your sock! If we have two variables $x$ and $y$, there are four unordered combinations of degree three:

$$x^3, \quad x^2y, \quad xy^2, \quad y^3.$$ 

But if order matters, there are eight:

$$xxx, xxx, xxy, xyx, yxx, yxy, yyx, yyy.$$ 

How can we count these ordered combinations? Here’s an approach based on the “symbolic series” method of the text. Let $S$ be the sum of all the ordered monomials in two variables $x$ and $y$ (and we throw in 1 for the empty monomial):

$$S = 1 + x + y + xx + xy + yx + yy + xxx + xxy + xyx + yxx + yyy + yxy + yyx + yyy + \cdots$$

Now since the order matters, every monomial (other than 1) must start with either $x$ or $y$. If we group these two sets of terms, we have

$$S = 1 + x(1 + x + y + xx + xy + yx + yy + \cdots) + \quad y(1 + x + y + xx + xy + yx + yy + \cdots),$$

or $S = 1 + xS + yS$. We solve this formally to get

(1) $$S = \frac{1}{1 - x - y},$$

whatever that means. Let’s try to make sense out of equation (1) by replacing each monomial $m$ by $t^{|m|}$, where $|m|$ is the degree of $m$. Then $S$ becomes the generating function $S(t)$ that counts ordered monomials by degree, and equation (1) becomes

$$S(t) = \frac{1}{1 - t - t} = \frac{1}{1 - 2t} = 1 + 2t + 4t^2 + 8t^3 + \cdots$$

Clearly the coefficient of $t^n$ is $2^n$, so there are $2^n$ ordered monomials of degree $n$ in $x$ and $y$ (or to put it another way, there are $2^n$ monomials of degree $n$ in noncommuting variables $x$ and $y$). This is really pretty obvious: in a monomial of degree $n$, you have $n$ factors and two choices ($x$ or $y$) for each factor. (Why doesn’t this reasoning work if $x$ and $y$ are allowed to commute?)
Exercise 1. Recall from the previous set of notes that there are \( \binom{n+2}{2} \) monomials of degree \( n \) in three commuting variables \( x, y, \) and \( z \). How many distinct monomials of degree \( n \) are there if \( x, y, \) and \( z \) don’t commute?

As the example of monomials shows, it’s actually easier to count things when you keep track of the order. Here’s another example. The ordered version of a partition is called a composition. Thus, \( 3 + 1 \) and \( 1 + 3 \) are considered distinct compositions of \( 4 \) (even though they represent the same partition). If we leave out the plus signs, we can write the eight compositions of \( 4 \) as

\[
(1111), \ (211), \ (121), \ (112), \ (22), \ (31), \ (13), \ (4).
\]

Let \( C \) be the symbolic sum of all compositions (with \( ( ) \) as the empty composition of \( 0 \)). Then

\[
C = ( ) + (1) + (11) + (2) + (111) + (21) + (12) + (3) + (1111) + (211) + (121) + (112) + (22) + (31) + (13) + (4) + \cdots
\]

Now every composition must start with 1, or 2, or 3, etc. So if we define “multiplication” of compositions by juxtaposition (e.g., \( (2) \star (11) = (211) \)), then

\[
C = ( ) \star C + (1) \star C + (2) \star C + (3) \star C + \cdots
\]

or formally

\[
C = \frac{( )}{( ) - (1) - (2) - (3) - \cdots}.
\]

Let’s try to make sense of equation (2) as we did with equation (1): replace each composition \( c \) by \( t^{|c|} \), where \( |c| \) is the weight of \( c \) (the sum of its parts). This takes \( C \) to the generating function \( C(t) \) that counts compositions by weight, so equation (2) becomes

\[
C(t) = \frac{1}{1 - t - t^2 - t^3 - \cdots} = \frac{1}{1 - \frac{t}{1-t}} = \frac{1-t}{1-2t}.
\]

Now

\[
\frac{1-t}{1-2t} = \frac{1}{1-2t} - \frac{t}{1-2t} = \sum_{n=0}^{\infty} 2^n t^n - \sum_{n=0}^{\infty} 2^n t^{n+1} = 1 + \sum_{n=1}^{\infty} (2^n - 2^{n-1}) t^n = 1 + \sum_{n=1}^{\infty} 2^{n-1} t^n,
\]

So there are \( 2^{n-1} \) compositions of \( n \).
Actually there is an easier way to see that $n$ has $2^{n-1}$ compositions: think of a row of $n$ dots. If you insert dividers into some of the $n - 1$ positions between the dots, you specify a composition of $n$. Since each of the $n - 1$ positions can have a divider or not, that’s $2^{n-1}$ choices. But the generating-function method is flexible enough to handle many related questions, such as the one in the next exercise.

**Exercise 2.** Let $Q_n$ be the number of compositions of $n$ in which all the parts are 1’s and 2’s. For example, $Q_5 = 8$ because there are eight such compositions of 5:

$$(11111), \ (2111), \ (1211), \ (1121), \ (1112), \ (221), \ (212), \ (122).$$

Find the generating function

$$Q(t) = 1 + \sum_{n \geq 1} Q_n t^n.$$ 

Does this look like a generating function we’ve seen before?

We can apply the same techniques to counting planar rooted trees. Let $P_n$ be the number of planar rooted trees with $n$ vertices. Then $P_4 = 5$ since there are five planar rooted trees with 4 vertices:

Now let’s let $\bar{F}_n$ be the number of ordered rooted forests of planar rooted trees. If we form a symbolic sum of all ordered forests, it looks like

$$\bar{F} = \emptyset + \bullet + \bullet + \cdots + \bullet + \bullet + \cdots + \bullet + \cdots$$

Since every nonempty ordered forest has a first tree, we can write this as

$$\bar{F} = \emptyset + \bullet \bar{F} + \bullet \bar{F} + \bullet \bar{F} + \bullet \bar{F} + \cdots$$

or

$$\bar{F} = \frac{\emptyset}{\emptyset} - \bullet - \bullet - \bullet - \cdots.$$ 

As with equations (1) and (2), we interpret equation (3) by replacing each symbol with $t^w$, where $w$ is the symbol’s weight (in this case, the number of vertices). Then equation (3) becomes

$$1 + \bar{F}_1 t + \bar{F}_2 t^2 + \cdots = \frac{1}{1 - P_1 t - P_2 t^2 - P_3 t^3 - \cdots}.$$
But every ordered forest of planar rooted trees with \( n \) vertices corresponds to a planar rooted tree with \( n + 1 \) vertices, so \( \tilde{F}_n = P_{n+1} \) and the preceding equation is

\[
(4) \quad 1 + P_2 t + P_3 t^2 + P_4 t^3 = \frac{1}{1 - P_1 t - P_2 t^2 - P_3 t^3 - \cdots}.
\]

This is much easier than the equation we had in the previous set of notes. For if we let \( P(t) = 1 + P_1 t + P_2 t^2 + \cdots \), then equation (4) multiplied by \( t \) is

\[
P(t) - 1 = \frac{t}{2 - P(t)}
\]

or

\[
P(t)^2 - 3P(t) + t + 2 = 0.
\]

Solve this using the quadratic formula to get

\[
P(t) = \frac{3 - \sqrt{9 - 4(t + 2)}}{2} = \frac{3 - \sqrt{1 - 4t}}{2},
\]

where we have chosen the negative square root to get \( P(0) = 1 \). But this says

\[
P(t) = 1 + \frac{1 - \sqrt{1 - 4t}}{2},
\]

and comparison with the generating function for Catalan numbers shows \( P_n = C_{n-1} \) for \( n \geq 1 \). Of course this is the same result we had earlier via our isomorphism of planar trees with parenthesized products.