COUNTING LABELLED COMBINATIONS

Often, we are interested in counting objects with labels. For example, there are 3 labelled trees with three vertices:

\[ \begin{array}{ccc}
1 & 2 & 3 \\
2 & 1 & 3 \\
2 & 3 & 1 \\
\end{array} \]

Similarly, there are 16 labelled trees with four vertices.

As in the case of unlabelled counting, we will consider “molecules” built from “atoms,” with each atom having a weight and the weight of a molecule being the sum of the weights of an atom. We think of an atom of weight \( n \) as being able to accept \( n \) labels: a labelled atom of weight \( n \) is an atom of weight \( n \) labelled by the set \( \{1, 2, \ldots, n\} \). A labelled molecule of weight \( n \) is a collection of atoms labelled by disjoint subsets of \( \{1, 2, \ldots, n\} \). For example, if the atoms are labelled trees, there are 7 molecules of weight three:

\[ \begin{array}{ccc}
1 & 2 & 3 \\
2 & 1 & 3 \\
3 & 2 & 1 \\
3 & 1 & 2 \\
\end{array} \]

If we know how to count labelled atoms, can we count labelled molecules? Let \( \hat{A}(t) \) be the exponential generating function of (nonempty) labelled atoms

\[ \hat{A}(t) = a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} + \cdots \]

and let \( \hat{M}(t) \) be the exponential generating function of labelled molecules. We shall prove the following “exponential formula”:

\[ (1) \quad \hat{M}(t) = \exp(\hat{A}(t)). \]

As in the unlabelled case, we will do this by stages. Suppose first that there is only one kind of labelled atom, of weight \( n \). Then \( \hat{A}(t) \) has only one term:

\[ \hat{A}(t) = \frac{t^n}{n!} \]

Now how many labelled molecules are there? There is just one labelled molecule of weight \( n \), the one-atom molecule. The next heavier molecule has weight \( 2n \), and
must contain exactly two atoms. But we have choices in how to split up the label set \{1, 2, \ldots, 2n\}: more precisely, we have
\[
\frac{1}{2} \binom{2n}{n} = \frac{(2n)!}{2(n!)^2}
\]
choices. This is because there are \( \binom{2n}{n} \) ways to choose the \( n \) labels for the first atom, leaving \( n \) labels for the second: but its doesn’t really matter which atom is first or second, hence the division by 2. As for molecules of weight 3\( n \), we can choose the \( n \) labels for the first atom in \( \binom{3n}{n} \) ways, leaving \( \binom{2n}{n} \) choices for the \( n \) labels on the second, which also determines the labels of the third. But it doesn’t really matter which atom is first, second, or third, so we divide by 6:
\[
\frac{1}{6} \binom{3n}{n} \binom{2n}{n} = \frac{(3n)!}{6(n!)^3}.
\]
Thus, our exponential generating function for molecules is
\[
\hat{M}(t) = 1 + \frac{t^n}{n!} + \frac{(2n)!}{2(n!)^2} \frac{t^{2n}}{(2n)!} + \frac{(3n)!}{6(n!)^3} \frac{t^{3n}}{(3n)!} + \cdots
\]
\[
= 1 + \frac{t^n}{n!} + \frac{1}{2} \left( \frac{t^n}{n!} \right)^2 + \frac{1}{6} \left( \frac{t^n}{n!} \right)^3 + \cdots
\]
\[
= \exp \left( \frac{t^n}{n!} \right)
\]
and equation (1) holds in this case.

Now suppose we have two kinds of atoms, “green” and “red.” Let
\[
\hat{M}_G(t) = 1 + g_1 t + g_2 \frac{t^2}{2!} + \cdots
\]
and
\[
\hat{M}_R(t) = 1 + r_1 t + r_2 \frac{t^2}{2!} + \cdots
\]
be the exponential generating functions for labelled molecules built exclusively of green and red molecules respectively. How many ways can we make a multicolored molecule of weight \( n \)? For any \( 0 \leq k \leq n \), we can combine a weight-\( k \) green molecule and a weight-(\( n - k \)) red molecule. We can choose the green molecule in \( g_k \) ways, the red molecule in \( r_{n-k} \) ways, and we can split up the label set \( \{1, 2, \ldots, n\} \) in \( \binom{n}{k} \) ways (the number of ways of picking the labels for the green molecule). So in all we can make our multicolored weight-\( n \) molecule in
\[
r_n + \binom{n}{1} g_1 r_{n-1} + \binom{n}{2} g_2 r_{n-2} + \cdots + g_n = \sum_{k=0}^{n} \binom{n}{k} g_k r_{n-k}
\]
ways: but this means that the exponential generating function \( \hat{M}(t) \) for multicolored molecules is \( \hat{M}_G(t)\hat{M}_R(t) \).

Now we're ready to prove equation (1) in general. First suppose we have \( a_n \) atoms of weight \( n \) (and no atoms of any other weight). The result of the previous paragraph says we should multiply the exponential generating functions for each type of atom together:

\[
\left( \exp \left( \frac{t^n}{n!} \right) \right)^{a_n} = \exp \left( a_n \frac{t^n}{n!} \right).
\]

Now let’s combine atoms of different weights. Again we multiply the generating functions:

\[
\hat{M}(t) = \exp(a_1 t) \exp \left( a_2 \frac{t^2}{2!} \right) \exp \left( a_3 \frac{t^3}{3!} \right) \cdots = \exp \left( a_1 t + a_2 \frac{t^2}{2!} + a_3 \frac{t^3}{3!} + \cdots \right).
\]

But this is equation (1).

Perhaps the simplest application of equation (1) is to count set partitions. Here the only labelled atom of weight \( n \) is the set \( \{1, 2, \ldots, n\} \), and a molecule of weight \( n \) is a partition of \( \{1, 2, \ldots, n\} \). Since \( a_i = 1 \) for \( i = 1, 2, \ldots, \), in this case, the exponential generating function for counting partitions is

\[
\hat{M}(t) = \exp \left( t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \right) = \exp(e^t - 1),
\]

a result we have already proved by other methods. But we can now do a lot more. For example, let \( E_n \) be the number of partitions of \( \{1, 2, \ldots, n\} \) into subsets with an even number of elements. What is the exponential generating function

\[
\hat{E}(t) = 1 + \sum_{n \geq 1} E_n \frac{t^n}{n!}?
\]

Since now we are only allowing atoms of even weight, equation (1) gives

\[
\hat{E}(t) = \exp \left( \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \right) = \exp(\cosh t - 1).
\]

Exercise 1. Work out a formula for the exponential generating function for \( O_n \), the number of partitions of \( \{1, 2, \ldots, n\} \) into subsets with an odd number of parts.

As another example of equation (1), suppose we want to count permutations. Each permutation can be written as a product of disjoint cycles: so a permutation is just a molecule, and a labelled atom of weight \( n \) is a cycle of \( \{1, 2, \ldots, n\} \). There are \( (n-1)! \) labelled atoms of weight \( n \): for example, the \( 3! = 6 \) cycles of \( \{1, 2, 3, 4\} \) are

\[
[1, 2, 3, 4], [1, 2, 4, 3], [1, 3, 2, 4], [1, 3, 4, 2], [1, 4, 2, 3], [1, 4, 3, 2].
\]

Thus, the exponential generating function for atoms is

\[
t + \frac{t^2}{2} + \frac{2t^3}{6} + \cdots + \frac{(n-1)! t^n}{n!} + \cdots = t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots + \frac{t^n}{n} + \cdots.
\]
Now from integrating the geometric series

\[ 1 + t + t^2 + t^3 + \cdots = \frac{1}{1 - t} \]

we have

\[ t + \frac{t^2}{2} + \frac{t^3}{3} + \cdots = -\log(1 - t) \]

and so the exponential generating function for permutations is

\[ \exp(-\log(1 - t)) = \frac{1}{1 - t} = 1 + t + t^2 + t^3 + \cdots \]

This is certainly right: the coefficient of \( t^n/n! \) in (2) is \( n! \), which is indeed the number of permutations of \( \{1, 2, \ldots, n\} \). But it hardly seems worth the effort—we already knew there are \( n! \) permutations of \( \{1, 2, \ldots, n\} \). The real power of the generating-function method, though, is that we can now solve a host of related problems. For example, how many derangements \( d_n \) of \( \{1, 2, \ldots, n\} \) are there? (Recall that a derangement is a permutation with no fixed points.) This is just like counting permutations, except that we don’t allow cycles of length 1. So our exponential generating function for atoms is now

\[ \frac{t^2}{2} + \frac{t^3}{3} + \cdots = -\log(1 - t) - t \]

and the exponential generating function for derangements is

\[ 1 + \sum_{n \geq 1} d_n \frac{t^n}{n!} = \exp(-\log(1 - t) - t) = \frac{e^{-t}}{1 - t} \]

Since

\[ e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots \]

and multiplying by \( (1 - t)^{-1} \) replaces the coefficient of \( t^n \) with the sum of the coefficients of \( 1, t, \ldots, t^{n-1}, t^n \), we have

\[ \frac{e^{-t}}{1 - t} = 1 + (1 - 1)t + (1 - 1 + \frac{1}{2!})t^2 + (1 - 1 + \frac{1}{2!} - \frac{1}{3!})t^3 + \cdots \]

and so equation (3) implies

\[ \frac{d_n}{n!} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}, \]

which is a result we obtained earlier by binomial inversion.

Here is another question: how many permutations \( od_n \) of \( \{1, 2, \ldots, n\} \) involve only cycles of odd length? Here we are allowing only atoms of odd weight, so our exponential generating function for atoms is
\[ t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots. \]

Observe that we get the same series from integrating

\[ 1 + t^2 + t^4 + t^6 + \cdots = \frac{1}{1-t^2}, \]

so

\[ t + \frac{t^3}{3} + \frac{t^5}{5} + \cdots = \int_0^t \frac{ds}{1-s^2} = \frac{1}{2} \log \left( \frac{1+t}{1-t} \right). \]

Now apply equation (1) to conclude that the exponential generating function for the odd-cycle permutations is

\[ 1 + \sum_{n \geq 1} od_n \frac{t^n}{n!} = \exp \left( \frac{1}{2} \log \left( \frac{1+t}{1-t} \right) \right) = \sqrt{\frac{1+t}{1-t}}. \]

Exercise 2. Show that the exponential generating function for the number of permutations \( ev_n \) of \( \{1, 2, \ldots, n\} \) involving only cycles of even length is

\[ 1 + \sum_{n \geq 1} ev_n \frac{t^n}{n!} = \frac{1}{\sqrt{1-t^2}}. \]