Combinatorics and Multiple Zeta Values

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Multiple Zeta Values

For positive integers $i_1, i_2, \ldots, i_k$ with $i_1 > 1$ (to ensure convergence), we define the multiple zeta value (henceforth MZV) $\zeta(i_1, i_2, \ldots, i_k)$ as the sum of the $k$-fold series

$$\sum_{n_1 > n_2 \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

We call $k$ the depth of $\zeta(i_1, \ldots, i_k)$, and $i_1 + \cdots + i_k$ its weight. Of course, the MZVs of depth one are the values $\zeta(i)$ of the Riemann zeta function at positive integers $i > 1$, for which Euler proved the formula

$$\zeta(2i) = \frac{(2\pi)^{2i}}{2(2i)!} |B_{2i}|,$$

but he also studied depth-two MZVs. Interest in MZVs of general depth dates from about 1990, stimulated by their appearance in physics and knot theory.
An important point is that the product of two MZVs is a sum of MZVs. For example,

\[
\zeta(2, 1)\zeta(2) = \sum_{i>j \geq 1} \frac{1}{i^2 j} \sum_{k \geq 1} \frac{1}{k^2} = \sum_{k>i>j \geq 1} \frac{1}{k^2 i^2 j} + \\
\sum_{i>j \geq 1} \frac{1}{i^4 j} + \sum_{i>k>j \geq 1} \frac{1}{i^2 k^2 j} + \sum_{i>j \geq 1} \frac{1}{i^2 j^3} + \sum_{i>j>k \geq 1} \frac{1}{i^2 j k^2}
\]

\[
= \zeta(2, 2, 1) + \zeta(4, 1) + \zeta(2, 2, 1) + \zeta(2, 3) + \zeta(2, 1, 2) \\
= 2\zeta(2, 2, 1) + \zeta(2, 1, 2) + \zeta(4, 1) + \zeta(2, 3)
\]

My study of this product led me to the quasi-symmetric functions, first introduced by Ira Gessel in 1984 as a source of generating functions for poset partitions.
Symmetric and Quasi-Symmetric Functions

Let $x_1, x_2, \ldots$ be a countable sequence of indeterminates, each of degree 1, and let

\[ \mathcal{P} \subset \mathbb{Q}[[x_1, x_2, \ldots]] \]

be the set of formal power series in the $x_i$ of bounded degree: $\mathcal{P}$ is a graded $\mathbb{Q}$-algebra. Any $f \in \mathcal{P}$ is quasi-symmetric if the coefficients in $f$ of

\[ x_{i_1}^{p_1} x_{i_2}^{p_2} \cdots x_{i_n}^{p_n} \quad \text{and} \quad x_{j_1}^{p_1} x_{j_2}^{p_2} \cdots x_{j_n}^{p_n} \]

agree whenever $i_1 < \cdots < i_n$ and $j_1 < \cdots < j_n$. The quasi-symmetric functions $\text{QSym}$ form an algebra, which properly includes the algebra $\text{Sym}$ of symmetric functions, e.g.,

\[ \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots \quad (2) \]

is quasi-symmetric but not symmetric.
Monomial (Quasi)symmetric Functions

For a composition (ordered sequence of positive integers) \( I = (i_1, \ldots, i_k) \), the corresponding monomial quasi-symmetric function \( M_I \in \text{QSym} \) is defined by

\[
M_I = \sum_{n_1 < n_2 < \cdots < n_k} x_{n_1}^{i_1} x_{n_2}^{i_2} \cdots x_{n_k}^{i_k}
\]

(so (2) above is \( M_{(2,1)} \)). Evidently \( \{M_I | I \text{ is a composition} \} \) is an integral basis for \( \text{QSym} \). For any composition \( I \), let \( \pi(I) \) be the partition given by forgetting the ordering. For any partition \( \lambda \), the monomial symmetric function \( m_\lambda \) is the sum of the \( M_I \) with \( \pi(I) = \lambda \), e.g.,

\[
m_{21} = M_{(2,1)} + M_{(1,2)}.
\]

The set \( \{m_\lambda | \lambda \text{ is a partition} \} \) is an integral basis for \( \text{Sym} \).
Other Bases for Sym

1. The **elementary** symmetric functions are

   \[ e_k = M_{(1)^k} = m_{\pi((1)^k)}, \]

   where \((1)^k\) is the composition consisting of \(k\) 1’s. Then \(\{e_\lambda | \lambda \text{ is a partition}\}\) is an integral basis for Sym, where \(e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots \) for \(\lambda = \pi(\lambda_1, \lambda_2, \ldots)\).

2. The **complete** symmetric functions are

   \[ h_k = \sum_{|I|=k} M_I = \sum_{|\lambda|=k} m_\lambda. \]

   Then \(\{h_\lambda | \lambda \text{ is a partition}\}\) is an integral basis for Sym.

3. The **power-sum** symmetric functions are

   \[ p_k = M_{(k)} = m_k. \]

   Then \(\{p_\lambda | \lambda \text{ is a partition}\}\) is a rational basis for Sym.
Multiplication of Monomial Quasi-Symmetric Functions

Two monomial quasi-symmetric functions $M_I$ and $M_J$ multiply according to a “quasi-shuffle” rule in which the parts of $I$ and $J$ are shuffled and also combined. For example,

\[ M_{(1)} M_{(1,2)} = M_{(1,1,2)} + M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,2)} + M_{(1,3)} \]

\[ = 2M_{(1,1,2)} + M_{(1,2,1)} + M_{(2,2)} + M_{(1,3)}. \]

In general the number of terms in the product $M_I M_J$ is

\[ \min \{ \ell(I), \ell(J) \} \sum_{i=0}^{\min \{ \ell(I), \ell(J) \}} \binom{\ell(I) + \ell(J) - i}{i, \ell(I) - i, \ell(J) - i} \]

so, e.g., $M_{(1,1)} M_{(2,1)}$ has \( \binom{4}{2} + \binom{3}{111} + \binom{2}{2} = 13 \) terms:

\[ M_{(1,1)} M_{(1,2)} = 3M_{(1,1,1,2)} + 2M_{(1,1,2,1)} + M_{(1,2,1,1)} \]

\[ + M_{(1,2,2)} + M_{(2,1,2)} + M_{(2,2,1)} + 2M_{(1,1,3)} + M_{(1,1,3,1)} + M_{(2,3)}. \]
Algebraic Structure of QSym

Over the rationals, the algebraic structure of QSym can be described as follows. Order compositions lexicographically:

\[(1) < (1, 1) < (1, 1, 1) < \cdots < (1, 2) < \cdots < (2) < (2, 1) < \cdots < (3) < \cdots\]

A composition \( I \) is Lyndon if \( I < K \) whenever \( I = JK \) for nonempty compositions \( J, K \).

**Theorem (Reutenauer-Malevenuto, 1995)**

QSym is a polynomial algebra on \( M_I, I \) Lyndon.

Since the only Lyndon composition ending in 1 is (1) itself, the subspace QSym\(^0\) of QSym generated by the \( M_I \) such that \( I \) ends in an integer greater than 1 is a subalgebra, and in fact QSym = QSym\(^0\)[\( M_{(1)} \)].
There is a homomorphism $\zeta : \text{QSym}^0 \to \mathbb{R}$ given by

$$\zeta(M_I) = \zeta(i_k, i_{k-1}, \ldots, i_1)$$

for $I = (i_1, \ldots, i_k)$, induced by sending $x_j \to \frac{1}{j}$. (This is well-defined since $i_k > 1$ for $M_I \in \text{QSym}^0$.) Note $\zeta(p_i) = \zeta(M_{(i)}) = \zeta(i)$ for $i > 1$. The intersection

$$\text{Sym}^0 = \text{QSym}^0 \cap \text{Sym}$$

is the subspace of Sym generated by the $m_\lambda$ such that all parts of $\lambda$ are 2 or more, and is in fact the subalgebra of Sym generated by the $p_i$ with $i > 1$. 
Knowing MZVs are homomorphic images has immediate consequences. For example, we know that the power-sums $p_n$ generate Sym rationally, so there is a polynomial $P_n$ with rational coefficients such that

$$e_n = P_n(p_1, p_2, \ldots, p_n).$$

Now apply to both sides the homomorphism sending $M(i_1, \ldots, i_m)$ to $M(ki_1, \ldots, ki_m)$ to get

$$M(k, \ldots, k) = P_n(p_k, p_{2k}, \ldots, p_{nk})$$

and thus

$$\zeta(k, \ldots, k) = P_n(\zeta(k), \zeta(2k), \ldots, \zeta(nk))$$
for any $k > 1$. In particular, since we know from equation (1) that $\zeta(k)$ is a rational multiple of $\pi^k$ if $k$ is even, and since $P_n$ has degree $n$, we have the following result.

**Theorem**

*If $k$ is even, then $\zeta(k, \ldots, k)$ is a rational multiple of $\pi^{kn}$.*

For particular values of $k$ more specific results are known, e.g.,

\[
\zeta(2, \ldots, 2) = \frac{\pi^{2n}}{(2n + 1)!}, \quad \zeta(4, \ldots, 4) = \frac{2^{2n+1}\pi^{4n}}{(4n + 2)!},
\]

\[
\zeta(6, \ldots, 6) = \frac{6(2\pi)^{6n}}{(6n + 3)!}.
\]
The Function $A_- : \text{QSym} \to \text{QSym}$

Consider the linear function from QSym to itself defined by

$$A_-(M(a_1,\ldots,a_{k-1},a_k)) = \begin{cases} M(a_1,\ldots,a_{k-1}), & \text{if } a_k = 1, \\ 0, & \text{otherwise}. \end{cases}$$

(Here $M_\emptyset$ is interpreted as 1, so $A_-(M(1)) = 1$.) From the quasi-shuffle description of multiplication in QSym the following result can be proved.

**Proposition**

$A_- : \text{QSym} \to \text{QSym}$ is a derivation.

Note that $\ker A_- = \text{QSym}^0$. So if we think of QSym as $\text{QSym}^0[M(1)]$, then $A_-$ is differentiation by $M(1)$. 
Extending $\zeta$ to $\zeta_u : \text{QSym} \rightarrow \mathbb{R}[u]$

Since $\text{QSym} = \text{QSym}^0[M(1)]$, we can extend $\zeta$ to a homomorphism $\zeta_u : \text{QSym} \rightarrow \mathbb{R}[u]$ by defining $\zeta_u(w) = \zeta(w)$ for $w \in \text{QSym}^0$ and $\zeta_u(M(1)) = u$. In view of the Proposition above, we have the following commutative diagram:

$$
\begin{array}{ccc}
\text{QSym} & \xrightarrow{\zeta_u} & \mathbb{R}[u] \\
A \downarrow & & \downarrow \frac{d}{du} \\
\text{QSym} & \xrightarrow{\zeta_u} & \mathbb{R}[u]
\end{array}
$$

In the case $u = \gamma \approx 0.5572$ (Euler’s constant), the homomorphism $\zeta_u$ turns out to be especially useful. To explain why we must digress a bit to introduce some generating functions.
Generating Functions

1. The generating function of the **elementary** symmetric functions is

\[ E(t) = \sum_{i \geq 0} e_i t^i = \prod_{i \geq 1} (1 + tx_i). \]

2. That for the **complete** symmetric functions is

\[ H(t) = \sum_{i \geq 0} h_i t^i = \prod_{i \geq 1} \frac{1}{1 - tx_i} = E(-t)^{-1}. \]

3. The logarithmic derivative of \( H(t) \) is the generating function \( P(t) \) for the **power sums**:

\[ P(t) = \frac{d}{dt} \log H(t) = - \sum_{i \geq 1} \frac{d}{dt} \log(1 - tx_i) = \sum_{i \geq 1} p_i t^{i-1}. \]
The power series expansion of $\psi(1 - t)$, where $\psi$ is the logarithmic derivative of the gamma function, is

$$\psi(1 - t) = -\gamma - \sum_{i \geq 2} \zeta(i) t^{i-1}.$$ 

If $u = \gamma$, then $\zeta_u(p_1) = \gamma$ and so $\zeta_u(P(t)) = -\psi(1 - t)$. Hence (since $P(t) = \frac{d}{dt} \log H(t)$) we have

**Theorem (Generating Function)**

$$\zeta_{\gamma}(H(t)) = \Gamma(1 - t).$$

From $E(t) = H(-t)^{-1}$ follows $\zeta_{\gamma}(E(t)) = \Gamma(1 + t)^{-1}$. 
Now it can be shown analytically that the generating function

\[ F(s, t) = \sum_{n,m \geq 1} \zeta(n+1, (1)^{m-1}) s^n t^m \]

can be written

\[ F(s, t) = 1 - \frac{\Gamma(1 - t)\Gamma(1 - s)}{\Gamma(1 - t - s)}. \]

Applying the Generating Function Theorem, this is

\[ F(s, t) = \zeta_\gamma \left( 1 - \frac{H(t)H(s)}{H(t+s)} \right). \]

Since

\[ H(t) = \exp \left( \sum_{i \geq 1} \frac{p_i t^i}{i} \right) = e^{p_1 t} \exp \left( \sum_{i \geq 2} \frac{p_i t^i}{i} \right) \]
it follows that

\[ 1 - \frac{H(t)H(s)}{H(t+s)} = 1 - \exp \left( \sum_{i \geq 2} p_i \frac{t^i + s^i - (t+s)^i}{i} \right) \]

and hence, applying \( \zeta_\gamma \),

\[ \sum_{n,m \geq 1} \zeta(n+1, (1)^{m-1}) s^n t^m = \]

\[ 1 - \exp \left( \sum_{i \geq 2} \zeta(i) \frac{t^i + s^i - (t+s)^i}{i} \right). \]

It follows that any MZV of the form \( \zeta(n+1, (1)^{m-1}) \) is a polynomial in the \( \zeta(i), i \geq 2 \), with rational coefficients, though \( M_{((1)^{m-1}, n+1)} \notin \text{Sym} \) for \( m > 1 \).
Multiple $t$-Values

Besides MZVs, other quantities are homomorphic images of quasi-symmetric functions. For a sequence $i_1, i_2, \ldots, i_k$ with $i_1 > 1$, the associated multiple $t$-value is

$$t(i_1, \ldots, i_k) = \sum_{n_1 > \cdots > n_k \geq 1, \; n_i \text{ odd}} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}.$$  

Evidently

$$t(i) = \sum_{n \geq 1 \; \text{odd}} \frac{1}{n^i} = (1 - 2^{-i})\zeta(i)$$

for $i \geq 2$, and from equation (1) it follows that

$$t(2i) = \frac{(-1)^{i-1}B_{2i}(2^{2i-1} - 1)\pi^{2i}}{2(2i)!}.$$
Then sending $x_j \to \frac{1}{2j-1}$ induces a homomorphism $\theta : \text{QSym}^0 \to \mathbb{R}$ with $\theta(M_{(i_1, \ldots, i_k)}) = t(i_1, \ldots, i_k)$ By essentially the same argument as for MZVs, we have that $t(k, \ldots, k)_{n}$

is a rational multiple of $\pi^{kn}$, but the multiple is different than for MZVs, e.g.,

$$t(2, \ldots, 2)_{n} = \frac{\pi^{2n}}{2^{2n}(2n)!}, \quad t(4, \ldots, 4)_{n} = \frac{\pi^{4n}}{2^{2n}(4n)!},$$

$$t(6, \ldots, 6)_{n} = \frac{\pi^{6n}}{8n(6n-1)!}.$$
We can also extend $\theta$ to a map from QSym to $\mathbb{R}$ by choosing a value for $\theta(M(1))$: the most natural choice seems to be $\log 2$. Then for the generating function $H(t)$ we have

$$\theta_{\log 2}(H(t)) = \frac{1}{\sqrt{\pi} e^{\gamma} t} \Gamma\left(1 - \frac{t}{2}\right).$$

Despite the similarities with MZVs, multiple $t$-values have a flavor all their own. For example, instead of $\zeta(2, 1) = \zeta(3)$ we have

$$t(2, 1) = -\frac{1}{2} t(3) + t(2) \log 2.$$
Another quantity one can consider are the finite truncations of MZV series:

\[ \zeta_{\leq n}(i_1, \ldots, i_k) = \sum_{n \geq n_1 \gg \cdots \gg n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}} \]

These are images of monomial quasi-symmetric functions via the mapping

\[ x_j \rightarrow \begin{cases} 
\frac{1}{j}, & j \leq n, \\
0, & \text{otherwise.} 
\end{cases} \]

There are no convergence issues here, so the homomorphism is immediately defined on all of QSym.
Now $\text{QSym}$ is a graded Hopf algebra, with coproduct

$$\Delta(M_I) = \sum_{I=JK} M_J \otimes M_K$$

where the sum is over all decompositions of $I$ as a juxtaposition (including the cases $J = \emptyset$ and $K = \emptyset$). Also, $\zeta_\gamma$ is a real character of the Hopf algebra $\text{QSym}$ (that is, an algebra homomorphism $\text{QSym} \to \mathbb{R}$). A character $\chi$ of $\text{QSym}$ is even if

$$\chi(w) = (-1)^{|w|} \chi(w)$$  \hspace{1cm} (3)

for homogeneous elements $w$ of $\text{QSym}$, and odd if

$$\chi(w) = (-1)^{|w|} \chi(S(w)),$$  \hspace{1cm} (4)

where $S$ is the antipode of $\text{QSym}$. 
The ABS Theorem

Theorem (Aguiar, Bergeron and Sottille, 2006)

Any real character of a (graded connected) Hopf algebra has a unique decomposition in the convolution algebra into an even character times an odd one.

Thus, there is an even character $\zeta_+$ and an odd character $\zeta_-$ so that $\zeta_\gamma = \zeta_+\zeta_-$ in the convolution algebra, i.e.,

$$\zeta_\gamma(w) = \sum_{(w)} \zeta_+(w_1)\zeta_-(w_2)$$

(5)

using Sweedler’s notation for coproducts:

$$\Delta(w) = \sum_{(w)} w_1 \otimes w_2.$$

\[\zeta : \text{QSym}^0 \rightarrow \mathbb{R} \text{ and its Extension } \zeta_u\]
Primitives

The power sums $p_i = M(i)$ are primitive, i.e.

$$\Delta(p_i) = 1 \otimes p_i + p_i \otimes 1,$$

and so

$$\zeta(i) = \zeta(p_i) = \zeta_-(p_i) + \zeta_+(p_i).$$

Equation (3) implies that $\zeta_+(w) = 0$ for any odd-dimensional $w$, particularly $w = p_i$, $i$ odd. On the other hand, equation (4) gives $\zeta_-(p_i) = 0$ for $i$ even, since $S(p_i) = -p_i$. Together with equation (6), this says

$$\zeta_+(p_i) = \begin{cases} 
\zeta(i), & i \text{ even}, \\
0, & i \text{ odd}, 
\end{cases} \quad \text{and} \quad \zeta_-(p_i) = \begin{cases} 
0, & i \text{ even}, \\
\zeta(i), & i > 1 \text{ odd}, \\
\gamma, & i = 1.
\end{cases}$$
Even Parts Theorem

In general, computing $\zeta_+(M_I)$ and $\zeta_-(M_I)$ for an arbitrary monomial quasi-symmetric function $M_I$ is difficult, apart from the general fact that

$$\zeta_+(M_I) = 0 \quad \text{if } |I| \text{ is odd.}$$

Nevertheless, there is the following result, which can be proved from the general theory of Aguiar-Bergeron-Sotille together with a specific result of Aguiar-Hsiao (2004).

**Theorem**

*If all parts of $I$ are even, then $\zeta_-(M_I) = 0$.***
Computing $\zeta_+$ and $\zeta_-$ on $\text{Sym}$

The situation is dramatically different if $\zeta_\gamma$ is restricted to $\text{Sym}$ (which is all we need for the $\Gamma$- and $\hat{\Gamma}$-genera). The $p_i$ generate $\text{Sym}$, so for every partition $\lambda$

$$m_\lambda = P_\lambda(p_1, p_2, p_3, p_4, \ldots)$$

for some polynomial $P_\lambda$ with rational coefficients. Since $\zeta_+$ and $\zeta_-$ are homomorphisms,

$$\zeta_-(m_\lambda) = P_\lambda(\gamma, 0, \zeta(3), 0, \ldots)$$

and

$$\zeta_+(m_\lambda) = P_\lambda(0, \zeta(2), 0, \zeta(4), \ldots).$$

In view of Euler's identity (1), the latter formula means that $\zeta_+(m_\lambda)$ is a rational multiple of $\pi^{|\lambda|}$ when $|\lambda|$ is even (and of course $\zeta_+(m_\lambda) = 0$ when $|\lambda|$ is odd).
Generating Functions Yet Again

Since the involution of Sym that exchanges the $e_i$ and the $h_i$ leaves the odd $p_i$ fixed, it follows that $\zeta_-(e_n) = \zeta_-(h_n)$, i.e.,

$$\zeta_\gamma \left( \frac{H(t)}{E(t)} \right) = 1.$$

On the other hand, $\zeta_+(E(t))$ is an even function, so $\zeta_+(E(t)) = \zeta_+(E(-t)) = \zeta_+(H(t))^{-1}$ and thus

$$\zeta_\gamma \left( \frac{H(t)}{E(t)} \right) = \zeta_+(H(t))^2.$$

Hence, using $\zeta_+ \zeta_- = \zeta_\gamma$ and the Generating Function Theorem,

$$\zeta_+(H(t))^2 = \zeta_+ \left( \frac{H(t)}{E(t)} \right) \zeta_- \left( \frac{H(t)}{E(t)} \right) = \zeta_\gamma \left( \frac{H(t)}{E(t)} \right) = \Gamma(1 - t) \Gamma(1 + t).$$
Because of the reflection formula for the gamma function, this is

$$\zeta_+(H(t))^2 = \frac{\pi t}{\sin \pi t}.$$ 

Thus we have the following result.

**Theorem (Factored Generating Function)**

The ABS factors of $\zeta_\gamma$ are given by

$$\zeta_+(E(t)) = \zeta_+(H(t))^{-1} = \sqrt{\frac{\sin \pi t}{\pi t}}$$

$$\zeta_-(E(t)) = \zeta_-(H(t)) = \sqrt{\frac{\sin \pi t}{\pi t}} \Gamma(1 - t).$$
Explicit Formula for $\zeta_-(e_n)$

There is also the explicit formula for $e_n$ in terms of power sums:

$$e_n = \sum_{i_1+2i_2+\cdots+ni_n=n} \frac{(-1)^n}{i_1!i_2!\cdots i_n!} (-p_1)^{i_1} \cdots (-\frac{p_n}{n})^{i_n},$$

which follows from

$$E(t) = H(-t)^{-1} = \exp \left( - \int_0^{-t} P(s) \, ds \right).$$

If we apply $\zeta_-$ to this, the minus signs cancel nicely to give

$$\zeta_-(e_n) = \sum_{i_1+3i_3+5i_5=\cdots=n} \frac{\gamma^{i_1} \zeta(3)^{i_3} \zeta(5)^{i_5} \cdots}{i_1!3^{i_3}i_3!5^{i_5}i_5!\cdots}. \quad (7)$$
An Example

To see how to put these results together, we note that

$$\sqrt{\frac{\sin \pi t}{\pi t}} = 1 - \frac{\pi^2 t^2}{12} + \frac{\pi^4 t^4}{1440} - \ldots$$

Then since the $e_i$ are divided powers,

$$\Delta(e_4) = 1 \otimes e_4 + e_1 \otimes e_3 + e_2 \otimes e_2 + e_3 \otimes e_1 + e_4 \otimes 1$$

and from the Factored Generating Function Theorem together with equations (5) and (7) it follows that

$$\zeta_\gamma(e_4) = \zeta_-(e_4) + \zeta_+(e_2)\zeta_-(e_2) + \zeta_+(e_4)$$

$$= \frac{\gamma^4}{4!} \cdot \frac{\gamma\zeta(3)}{3} - \frac{\pi^2}{12} \cdot \frac{\gamma^2}{2} + \frac{\pi^4}{1440}$$

$$= \frac{\gamma^4}{24} - \frac{\gamma^2\zeta(2)}{4} + \frac{\gamma\zeta(3)}{3} + \frac{\zeta(4)}{16}.$$
Factoring $\theta_{\log 2}$

Recall that $\theta_{\log 2} : \text{QSym} \to \mathbb{R}$ is the homomorphism that sends $M(1)$ to $\log 2$ and $M(i_1, ..., i_k)$ to the multiple $t$-value $t(i_k, \ldots, i_1)$ when $i_1 > 1$. Just as $\zeta_\gamma$ can be factored, so can $\theta_{\log 2}$. In this case the result is as follows.

**Theorem**

The ABS factorization $\theta_{\log 2} = \theta_+ \theta_-$ satisfies

$$\theta_+(H(t)) = \sqrt{\sec \frac{\pi t}{2}} = \theta_+(E(t))^{-1}.$$