Interaction of Combinatorics and Algebra: Updown Categories and Hopf Algebras

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Outline

1. Updown Categories
2. Hopf Algebras
3. Pieri Hopf Algebras
4. Pieri Hopf Algebras with $B_+$
Consider a partially ordered set $P$ with a natural grading

$$P = \bigcup_{i \geq 0} P_i$$

such that if $y \in P$ covers $x \in P$ (that is, $x \prec y$ and there is no $z$ with $x \prec z \prec y$), then $x$ and $y$ are in adjacent ranks, i.e., $x \in P_i$ and $y \in P_{i+1}$. We write $|x|$ for the rank of $x$. Assume each $P_i$ is finite and $P_0 = \{\hat{0}\}$, where $\hat{0}$ is the least element of $P$. We say $P$ is an updown category if each $x \in P$ has an associated (finite) automorphism group $\text{Aut}_x$, and for any $x, y \in P$ with $y$ covering $x$ there are positive integers $u(x; y)$ and $d(x; y)$ such that

$$d(x; y) | \text{Aut}_x| = u(x; y) | \text{Aut}_y|. \quad (1)$$
Now let $P$ be an updown category, $k$ a field of characteristic 0, and $kP$ the graded vector space with elements of $P$ as basis. We define operators $U$ and $D$ on $kP$ as follows:

$$Uc = \sum_{|c'| = |c| + 1} u(c; c')c'$$

for $c \in P$, and

$$Dc = \sum_{|c'| = |c| - 1} d(c'; c)c'$$

for $c \in P$, $c \neq \hat{0}$; we set $D\hat{0} = 0$. (Here $u(c; c') = d(c; c') = 0$ if $c'$ doesn’t cover $c$.)
Now define an inner product $\langle \cdot, \cdot \rangle$ on $kP$ by setting

$$\langle c, d \rangle = \begin{cases} \lvert \text{Aut}(c) \rvert, & \text{if } d = c, \\ 0, & \text{otherwise}. \end{cases}$$

Then if $\lvert y \rvert = \lvert x \rvert + 1$,

$$\langle U(x), y \rangle = \langle \sum u(x; x')x', y \rangle = u(x; y)\lvert \text{Aut} y \rvert$$

while

$$\langle x, D(y) \rangle = \langle x, \sum d(y'; y)y' \rangle = d(x; y)\lvert \text{Aut} x \rvert,$$

so $U$ and $D$ are adjoint by equation (1).
We can now extend the multiplicities $u(c; c')$ and $d(c; c')$ to all pairs of elements $c, c' \in P$: suppose $|c'| = |c| + k$. If $k < 0$ we put $u(c; c') = d(c; c') = 0$, while for $k \geq 0$

\[
    u(c; c') = \frac{\langle U^k(c), c' \rangle}{|\text{Aut}(c')|} = \frac{\langle c, D^k(c') \rangle}{|\text{Aut}(c')|}
\]

and

\[
    d(c; c') = \frac{\langle U^k(c), c' \rangle}{|\text{Aut}(c)|} = \frac{\langle c, D^k(c') \rangle}{|\text{Aut}(c)|}.
\]

Then equation (1) extends to arbitrary pairs of objects. In particular,

\[
    \frac{d(\hat{0}; c)}{u(\hat{0}; c)} = \frac{|\text{Aut}(c)|}{|\text{Aut}(\hat{0})|} \quad (2)
\]

for any $c \in P$. 
It is also easy to prove that

\[ u(c; c') = \sum_{|c''| = k} u(c; c'') u(c''; c') \]

for any \( |c| \leq k \leq |c'| \), and similarly for \( d \).

An updown category \( P \) is called univalent if \( u(c; c') = d(c; c') \)
for any pair \( c, c' \), and unital if \( u(c; c') = d(c; c') \) is either 0 or
1 for any \( c, c' \) with \( |c'| = |c| + 1 \). Unital implies univalent but
not conversely. In the univalent case (with \( \text{Aut} \hat{0} \) trivial) all the
automorphism groups are trivial, by equation (2).
If we write $U_i, D_i$ respectively for $U, D$ restricted to $kP_i$, then we can consider various conditions on the commutator

$$[D, U]_i = D_{i+1}U_i - U_{i-1}D_i.$$ 

1. Absolute commutation condition (ACC): $[D, U]_i = rl_i$, with $r$ independent of $i$.
2. Linear commutation condition (LCC): $[D, U]_i = (ai + b)l_i$.
3. Sequential commutation condition (SCC): $[D, U]_i = r_il_i$.
4. Weak commutation condition (WCC): $[D, U]_i$ is a diagonal matrix.
Evidently ACC $\implies$ LCC $\implies$ SCC $\implies$ WCC. The WCC amounts to the condition that there is a function $\epsilon$ on $P$ so that $[D, U]p = \epsilon(p)p$ for all $p \in P$. If $\epsilon(p)$ only depends on $|p|$, then $P$ satisfies the SCC. If $\epsilon$ is a linear function of $|p|$, then $P$ satisfies the LCC. If $\epsilon$ is a constant function, then $P$ satisfies the ACC. If $P$ is a chain (i.e., $|P_i| \leq 1$ for all $i$), then the SCC is automatic.

Stanley’s differential posets are unital updown categories satisfying the ACC.

Given two updown categories, we can form their product. Any product of updown categories with the ACC has the ACC, and similarly for the LCC and WCC. This is not true of the SCC.
Example: Finite Sets

As a first example of an updown category, consider the poset $S$ which is a chain with the sole element of $S_n$ being the set $\{1, 2, \ldots, n\}$, which we denote $[n]$ (put $[0] = \emptyset$). Set $u([n]; [n+1]) = 1$, $d([n]; [n+1]) = n + 1$, and let $\text{Aut}[n]$ be the symmetric group on $n$ symbols. Equation (1) is easily seen to hold, and since we have

$$(DU - UD)[n] = D[n+1] - Un[n-1] = (n+1)[n] - n[n] = [n]$$

$S$ satisfies the ACC.
Example: Partitions

Let $\mathcal{K}$ be the updown category such that $\mathcal{K}_n$ is the set of partitions of $n$. Writing a partition as $(\lambda_1, \ldots, \lambda_l)$, the rank is

$$|(\lambda_1, \ldots, \lambda_l)| = \lambda_1 + \cdots + \lambda_l.$$

For any partition $\lambda$, let $m_k(\lambda)$ be the number of parts of size $k$ in $\lambda$. If $|\mu| = |\lambda| + 1$, then $\mu$ covers $\lambda$ if (a) $\mu$ comes from $\lambda$ by adding a new part of size 1, or (b) $\mu$ comes from $\lambda$ by increasing a part of $\lambda$ in size. In case (a), $u(\lambda; \mu) = 1$ and $d(\lambda; \mu) = m_1(\mu)$. In case (b), $u(\lambda; \mu) = m_k(\lambda)$ and $d(\lambda; \mu) = m_{k+1}(\mu)$, where $\mu$ is gotten by increasing a size-$k$ part of $\lambda$ to size $k+1$. 
The automorphism group Aut(\(\lambda\)) exchanges parts of \(\lambda\) having the same size, so

\[ |\text{Aut}(\lambda)| = m_1(\lambda)! m_2(\lambda)! \cdots \]

It can be shown that

\[ [D, U] \lambda = (1 + m_1(\lambda))\lambda, \quad (3) \]

so \(K\) satisfies the WCC.
Example: Rooted Trees

Let $\mathcal{T}$ be the set of rooted trees, with rank function

$$|t| = \text{the number of non-root vertices of } t.$$

If $|t'| = |t| + 1$, then $t'$ covers $t$ if and only if a new edge and terminal vertex can be attached to $t$ to give $t'$. Let $u(t; t')$ be the number of vertices of $t$ to which a new edge and terminal vertex can be added to get $t'$, and $d(t; t')$ the number of terminal edges of $t'$ that, when deleted, leave $t$. For example,

$$u( \quad ; \quad ) = 1$$

while

$$d( \quad ; \quad ) = 2.$$
If $t$ is a rooted tree, the group $\text{Aut } t$ can be described as follows. Let $p$ be a vertex of $t$, and $q_1, \ldots, q_n$ its children: each $q_i$ can be considered the root of a rooted tree $t_i$. Let $G(p)$ be the group of permutations of the $q_i$ that exchange $q_i$ with $q_j$ only if $t_i$ and $t_j$ are isomorphic rooted trees. Then

$$\text{Aut } t = \prod_{\text{vertices } p \text{ of } t} G(p).$$

A simple argument shows that

$$[D, U]t = (|t| + 1)t,$$

(4)

so $\mathcal{T}$ satisfies the LCC.
Let $\mathcal{A}$ be a unital associative algebra over $k$. We assume $\mathcal{A}$ is graded, i.e.,

$$\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$$

with $\mathcal{A}_n \mathcal{A}_m \subset \mathcal{A}_{n+m}$. To say $\mathcal{A}$ is connected means that the degree 0 part of $\mathcal{A}$ consists of scalars, i.e., $\mathcal{A}_0 = k1$.

A coalgebra structure on $\mathcal{A}$ consists of homomorphisms $\eta : \mathcal{A} \to k$ (counit) and $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (coproduct), so that $\eta$ sends $1 \in \mathcal{A}$ to $1 \in k$ and all positive-degree elements of $\mathcal{A}$ to $0$, and $\Delta$ respects the grading. These satisfy

$$(\text{id}_{\mathcal{A}} \otimes \eta)\Delta = (\eta \otimes \text{id}_{\mathcal{A}})\Delta = \text{id}_{\mathcal{A}}.$$ (5)

We also assume $\Delta$ is coassociative, i.e., $\Delta(\Delta \otimes \text{id}_{\mathcal{A}}) = \Delta(\text{id}_{\mathcal{A}} \otimes \Delta)$. For $\mathcal{A}$ to be a Hopf algebra, $\Delta$ must be a homomorphism of graded algebras.
Properties of Hopf algebras

Writing the comultiplication applied to $u \in \mathcal{A}$ as

$$\Delta(u) = \sum_u u' \otimes u'',$$

we note that condition (5) requires that it have the form

$$u \otimes 1 + \sum_{|u'|, |u''| > 0} u' \otimes u'' + 1 \otimes u.$$

If $\Delta(u) = u \otimes 1 + 1 \otimes u$, then $u$ is primitive.

A Hopf algebra $\mathcal{A}$ has an antipode $S : \mathcal{A} \rightarrow \mathcal{A}$, which is an antiautomorphism of $\mathcal{A}$ with $S(1) = 1$ and

$$\sum_u S(u')u'' = \sum_u u'S(u'') = 0 \quad \text{for} \quad |u| > 0.$$

Hence $S(u) = -u$ if $u$ is primitive. If $\mathcal{A}$ is either commutative or cocommutative, $S^2 = \text{id}$. 
The polynomial algebra $k[t]$ can be made a graded connected Hopf algebra by declaring $t$ primitive. Then the coproduct in general is given by

$$\Delta(t^n) = \sum_{k=0}^{n} \binom{n}{k} t^k \otimes t^{n-k},$$

and the antipode by

$$S(t^n) = (-1)^n t^n.$$
A Hopf algebra structure of Sym can be specified by requiring that the elementary symmetric functions $e_i$ be divided powers, i.e.,

$$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i},$$

where it is understood that $e_0 = 1$. Then Sym is a graded connected Hopf algebra: the primitives are exactly the elements $m_k$ (the power sums). In fact,

$$\Delta(m_\lambda) = \sum_{\mu \cup \nu = \lambda} m_\mu \otimes m_\nu.$$ (7)

Since Sym is commutative, the antipode $S$ is an algebra isomorphism with $S^2 = \text{id}$. In fact, $S(e_k) = (-1)^k h_k$. 
To define a multiplication on $k\mathcal{T}$, let $t, t'$ be rooted trees, and suppose that $t$ is the result of adjoining a root to the the “forest” $t_1 t_2 \cdots t_n$ (written $t = B_+(t_1 \cdots t_n)$). There are $|t'|^n$ rooted trees obtainable by attaching each of the $n$ rooted trees $t_1, \ldots, t_n$ to some vertex of $t'$: let $t \ast t' \in k\mathcal{T}$ be the sum of these trees. (If $t = \bullet$, define $t \circ t'$ to be $t'$.) This defines a noncommutative multiplication

$$\ast : k\mathcal{T}_n \otimes k\mathcal{T}_m \to k\mathcal{T}_{n+m},$$

so $\ast$ makes $k\mathcal{T}$ a graded algebra with $\bullet \in k\mathcal{T}_0$ as two-sided identity.
Grossman-Larson Hopf Algebra cont’d

The coalgebra structure on $k\mathcal{I}$ is given by

$$
\Delta(t) = \sum_{I \cup J = \{1,2,\ldots,n\}} B_+(t_I) \otimes B_+(t_J),
$$

where $t = B_+(t_1 t_2 \cdots t_n)$ and the sum is over all pairs $(I, J)$ of (possibly empty) subsets $I, J$ of $\{1, \ldots, n\}$ such that $I \cup J = \{1, \ldots, n\}$; also, $t_I$ means the juxtaposition of $t_i$ for $i \in I$ (and $B_+(t_\emptyset) = \bullet$). Grossman and Larson proved that $k\mathcal{I}$ with the product $\ast$ and coproduct $\Delta$ is a graded Hopf algebra, which we henceforth denote $\mathcal{H}_{GL}$. 
We say an updown category $P$ has a Pieri algebra structure if the vector space $kP$ admits a graded algebra structure

$$kP_n \otimes kP_m \rightarrow kP_{n+m}$$

such that $\hat{0} \in P_0$ is the identity and

$$U(c) = U(\hat{0})c$$

for all $c \in P$. 
Examples

1. The updown category $S$ has a Pieri algebra structure obtained by identifying $[n]$ with $t^n$ in the polynomial algebra $k[t]$.

2. The updown category $\mathcal{K}$ has a Pieri algebra structure from identifying $\lambda \in \mathcal{K}$ with the normalized monomial symmetric function $\tilde{m}_\lambda$ in the algebra $\text{Sym}$ of symmetric functions, where $\tilde{m}_\lambda = |\text{Aut}\lambda| m_\lambda$.

3. The updown category $\mathcal{T}$ has a Pieri algebra structure given by the Grossman-Larson product.
In all the examples above, the Pieri algebra $kP$ is also a graded connected Hopf algebra: we call this a Pieri Hopf algebra structure. Using the inner product on $kP$, we have a second multiplication $\circ$ on $kP$ defined by

$$\langle u \circ v, w \rangle = \langle u \otimes v, \Delta(w) \rangle$$

where $\Delta$ is the coproduct.

**Theorem**

$D$ is a derivation for $\circ$, i.e., $D(u \circ v) = D(u) \circ v + u \circ D(v)$. 
If the dual of $\Delta$ coincides with the product, we have the following.

**Theorem**

*If $kP$ is a Pieri Hopf algebra such that $u \circ v = uv$, then $P$ satisfies the ACC.*

**Proof.**

For any $p \in P$,

$$[D, U]p = D(U(\hat{0}) \circ p) - U(\hat{0})D(p)$$

$$= DU(\hat{0}) \circ p + U(\hat{0}) \circ D(p) - U(\hat{0}) \circ D(p)$$

$$= DU(\hat{0}) \circ p$$
Pieri Hopf Algebras cont’d

This result applies to the case of $S$, since in $k[t]$ the inner product is

$$\langle t^n, t^m \rangle = n! \delta_{n,m}$$

and so, for $k + l = n$,

$$\langle t^k \circ t^l, t^n \rangle = \langle t^k \otimes t^l, \Delta(t^n) \rangle = \langle t^k \otimes t^l, \sum_{i+j=n} \binom{n}{i} t^i \otimes t^j \rangle = \sum_{i+j=n} \binom{n}{i} \langle t^k, t^i \rangle \langle t^l, t^j \rangle = \binom{n}{k} \binom{n}{l} = n! = \langle t^n, t^n \rangle.$$
Suppose $U$ satisfies

$$U(p \circ q) = U(p) \circ q + p \circ U(q) - p \circ U(\hat{0}) \circ q$$  \hspace{1cm} (8)

for all $p, q \in P$. Then we call $U$ a pre-derivation. Of course any derivation is a pre-derivation, and a computation establishes

**Proposition**

*The commutator of pre-derivations is a pre-derivation.*

This allows us to prove the following result.

**Theorem**

*Suppose $kP$ is a Pieri Hopf algebra and $(kP, \circ)$ is generated as an algebra by eigenvectors of $[D, U]$. The $P$ satisfies the WCC.*
Proof.

Let $p, q$ be eigenvectors of $[D, U]$ with eigenvalues $\epsilon(p), \epsilon(q)$ respectively. Then $[D, U]$ is a pre-derivation, so

$$[D, U](p \circ q) = [D, U]p \circ q + p \circ [D, U]q - p \circ [D, U]\hat{0} \circ q$$

$$= (\epsilon(p) + \epsilon(q) - \epsilon(\hat{0}))p \circ q$$

This result applies to $\mathcal{K}$, since in Sym equation (7) implies

$$\tilde{m}_\alpha \circ \tilde{m}_\beta = \tilde{m}_{\alpha \cup \beta},$$

and equation (8) follows. (In fact, equation (3) follows from the proof above.)
We say a Pieri Hopf algebra $kP$ admits a $B_+$ operator if there is an injective linear function $B_+ : kP \to kP$ that increases degree by 1 such that

$$\Delta^* B_+(u) = B_+(u) \otimes \hat{0} + (\text{id} \otimes B_+)\Delta^*(u)$$

for all $u \in kP$. Here $\Delta^*$ is the dual via the inner product of the coproduct $\Delta$.

**Theorem**

*If $kP$ admits a $B_+$ operator, then $DB_+(u) = B_+D(u)$ for $|u| > 0$.***
Two of our earlier examples of Pieri Hopf algebras admit $B_+$ operators:

1. The Pieri Hopf algebra $kS = k[t]$ has a $B_+$ operator given by

$$B_+(t^n) = \frac{t^{n+1}}{n+1}.$$ 

2. The Pieri Hopf algebra $kT = \mathcal{H}_{GL}$ has a $B_+$ operator that attaches a new root vertex and edge to the root of $t$:

$$B_+(\quad) = \quad.$$
Pieri Hopf Algebras with $B_+$

The following result distinguishes the first of our examples from the second (recall that $\mathcal{H}_{GL}$ is noncommutative).

**Theorem**

Suppose the Pieri Hopf algebra $kP$ admits a $B_+$ operator, $P_1$ has a single element, and the multiplication in $kP$ is commutative. Then there is a scalar $t$ with $B_+D(u) = DB_+(u) = tu$ for all $|u| > 0$. It follows that $P$ is an infinite chain.

This result also demonstrates that $\text{Sym} = kS$ cannot admit a $B_+$ operator.
Arboreal Structures

Let $kP$ be a Pieri Hopf algebra with $B_+$. We say $kP$ has a unilateral arboreal structure if

$$UB_+(p) = B_+ U(p) + U(\hat{0}) \circ B_+(p)$$

for all $p \in P$, and a bilateral arboreal structure if

$$UB_+(p) = B_+ U(p) + U(\hat{0}) \circ B_+(p) + B_+(p) \circ U(\hat{0}).$$

**Theorem**

Suppose $kP$ is a Pieri Hopf algebra with $B_+$ that has an arboreal structure such that $kP$ can be generated via $\circ$ and $B_+$ from eigenvectors of $[D, U]$. Then $P$ satisfies the WCC.
The Pieri Hopf algebra $k\mathcal{T} = \mathcal{H}_{GL}$ has a unilateral arboreal structure. In fact, $\mathcal{H}_{GL}$ is generated by $B_+$ and $\circ$ from the one-vertex tree $\bullet$. Thus $\mathcal{T}$ satisfies the WCC, and in fact the equation (4) that implies $\mathcal{T}$ satisfies the LCC follows by working through the proof.

There is an updown category $\mathcal{P}$ of planar rooted trees, which turns out to be univalent. Then $k\mathcal{P}$ has a Pieri Hopf algebra structure with $B_+$. This Hopf algebra has a bilateral arboreal structure. The preceding result can be used to show that $\mathcal{P}$ has the WCC, and in fact that

$$[D, U]t = (2|t| + \tau(t) + 1)t,$$

where $\tau(t)$ is the number of terminal vertices of $t$. 

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