Sums of Generalized Harmonic Series (NOT Multiple Zeta Values)

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Outline

1. Introduction

2. $H$-series and Stirling numbers of the first kind

3. Proof of the sum formulas
The generalized harmonic series we will be concerned with are

\[ H(a_1, a_2, \ldots, a_k) = \sum_{n=1}^{\infty} \frac{1}{n^{a_1}(n+1)^{a_2} \cdots (n+k-1)^{a_k}}, \]

where \( a_1, \ldots, a_k \) are nonnegative integers whose sum is at least 2. We call this quantity an \( H \)-series of length \( k \) and weight \( a_1 + \cdots + a_k \). Of course \( H \)-series of length 1 are just values of the Riemann zeta function. We will prove that

\[ \sum_{a_1+\cdots+a_k=m} H(a_1, \ldots, a_k) = k \zeta(m), \quad (1) \]

where \( m \geq 2 \) and the sum is over \( k \)-tuples of nonnegative integers. This was conjectured by C. Moen last spring, and proved by both of us over the summer.
First cases

Equation (1) is only interesting if $k > 1$. The first case is

$$H(2, 0) + H(1, 1) + H(0, 2) = 2\zeta(2),$$

which is pretty trivial: evidently $H(2, 0) = \zeta(2)$ and $H(0, 2) = \zeta(2) - 1$, and the usual telescoping-sum argument shows

$$H(1, 1) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

More generally, equation (1) for $k = 2$ is

$$H(m, 0) + H(m - 1, 1) + \cdots + H(1, m - 1) + H(0, m) = 2\zeta(m),$$

which reduces to

$$H(m - 1, 1) + \cdots + H(1, m - 1) = 1. \quad (2)$$
Multiple zeta values

Equation (1) is superficially similar to the “sum theorem” for multiple zeta values (conjectured by C. Moen 1988, proved by A. Granville 1997):

$$\sum_{a_1+\ldots+a_k=m, \ a_i \geq 1, \ a_1 \geq 2} \zeta(a_1, a_2, \ldots, a_k) = \zeta(m) \quad (3)$$

for $m \geq 2$, where

$$\zeta(a_1, a_2 \ldots, a_k) = \sum_{n_1>n_2>\ldots>n_k\geq 1} \frac{1}{n_1^{a_1} n_2^{a_2} \cdots n_k^{a_k}}.$$

But note that the multiple zeta value $\zeta(a_1, \ldots, a_k)$ is a $k$-fold sum, while $H(a_1, \ldots, a_k)$ is a simple sum. Also, the left-hand side of (3) is a sum over $k$-tuples of positive integers, while that of (1) is a sum over $k$-tuples of nonnegative integers.
Another sum of H-series

We could ask what happens if we sum $H$-series over all $k$-tuples of positive integers with sum $m \geq 2$. There is a nice formula in this case, though by no means as simple as equation (1):

$$
\sum_{a_1 + \cdots + a_k = m, \ a_i \geq 1} H(a_1, \ldots, a_k) =
$$

$$
\frac{1}{(k - 1)!} \sum_{i=0}^{k-2} \binom{k - 2}{i} \frac{(-1)^i}{(i + 1)^{m-k+1}}. \quad (4)
$$

Note that the right-hand side is a rational number. The case $k = 2$ is equation (2).
Telescopong

We shall build up a chain of lemmas on $H$-series. Using 
\[ \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}, \]
note that for $a, b \geq 1$,

\[ H(a, b) = \sum_{n=1}^{\infty} \frac{1}{n^a(n+1)^b} = \]

\[ \sum_{n=1}^{\infty} \left[ \frac{1}{n^a(n+1)^{b-1}} - \frac{1}{n^{a-1}(n+1)^b} \right] = H(a, b-1) - H(a-1, b). \]

Hence

\[ H(m - 1, 1) + H(m - 2, 2) + \cdots + H(1, m - 1) = \]

\[ H(m - 1, 0) - H(m - 2, 1) + H(m - 2, 1) - H(m - 3, 2) + \cdots + H(1, m - 2) - H(0, m - 1) = H(m - 1, 0) - H(0, m - 1) = 1, \]

proving equation (2).
Sum over indices of fixed sum, fixed positions

This generalizes as follows.

**Lemma (1)**

Let $1 \leq i < j \leq p$, and let $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_p$ be fixed nonnegative integers. Then

$$
\sum_{k=1}^{m-1} H(a_1, \ldots, a_{i-1}, k, a_{i+1}, \ldots, a_{j-1}, m-k, a_{j+1}, \ldots, a_p) = \\
\frac{1}{j-i} [H(a_1, \ldots, a_{i-1}, m-1, a_{i+1}, \ldots, a_{j-1}, 0, a_{j+1}, \ldots, a_p) \\
- H(a_1, \ldots, a_{i-1}, 0, a_{i+1}, \ldots, a_{j-1}, m-1, a_{j+1}, \ldots, a_p)].
$$
We can use Lemma (1) to obtain a formula for the sum of all $H$-series of fixed length and weight and nonzero entries at specified positions. Let $H_{p,q}^{(r)} = \sum_{j=p}^{q} j^{-r}$ for $p \leq q$.

**Lemma (2)**

For $k \geq 1$ and $1 \leq i_0 < i_1 < \cdots < i_k \leq n$, let $P(i_0, \ldots, i_k)$ be the set of strings $(a_1, \ldots, a_n)$ of nonnegative integers with $a_1 + \cdots + a_n = m$ and $a_j \neq 0$ iff $j = i_q$ for some $q$. Then

\[
\sum_{(a_1, \ldots, a_n) \in P(i_0, \ldots, i_k)} H(a_1, \ldots, a_n) = \sum_{j=1}^{k} (-1)^{j-1} H_{i_0, i_j-1}^{(m-k)} \left( \frac{(i_j - i_0)(i_j - i_1) \cdots (i_j - i_{j-1})(i_{j+1} - i_j) \cdots (i_k - i_j)}{(i_j - i_0)(i_j - i_1) \cdots (i_j - i_{j-1})(i_{j+1} - i_j) \cdots (i_k - i_j)}. \right)
\]
For $0 \leq k \leq n - 1$, let $C(k, n; m)$ be the sum of all $H$-series $H(a_1 \ldots, a_n)$ of length $n$, weight $m$, and exactly $k + 1$ of the $a_i$ nonzero. If $k \geq 1$, $C(k, n; m)$ is the sum over all sequences $1 \leq i_0 < \cdots < i_k \leq n$ of the sums given by the preceding result; since each $H_{i_0,i_j-1}^{(m-k)}$ in that result is a sum of $\frac{1}{p^{m-k}}$ with $1 \leq i_0 \leq p \leq i_j - 1 \leq n - 1$, we have

$$C(k, n; m) = \sum_{j=1}^{n-1} \frac{c^{(n)}_{k,j}}{j^{m-k}}$$

with $c^{(n)}_{k,j}$ rational. We can deduce results about the $c^{(n)}_{k,j}$ from Lemma (2).
An (anti)symmetry property

**Lemma (3)**

For $1 \leq k, j \leq n - 1$, $c_{k,n-j}^{(n)} = (-1)^{k-1} c_{k,j}^{(n)}$.

**Proof.**

For $1 \leq i_0 < \cdots < i_k \leq n$, let $S_n^m(i_0, \ldots, i_k)$ be the left-hand side of the equation in Lemma (2). Then if

$$S_n^m(i_0, \ldots, i_k) = p_1 + \frac{p_2}{2^{m-k}} + \cdots + \frac{p_{n-1}}{(n-1)^{m-k}},$$

it follows from Lemma (2) that

$$(-1)^{k-1} S_n^m(n + 1 - i_k, \ldots, n + 1 - i_0) =$$

$$p_{n-1} + \frac{p_{n-2}}{2^{m-k}} + \cdots + \frac{p_1}{(n-1)^{m-k}}.$$
Some explicit coefficients

Here are some of the coefficients $c_{k,j}^{(n)}$.

\[
\begin{align*}
  c_{1,1}^{(3)} &= \frac{3}{2} & c_{1,2}^{(3)} &= \frac{3}{2} \\
  c_{2,1}^{(3)} &= \frac{1}{2} & c_{2,2}^{(3)} &= -\frac{1}{2} \\
  c_{1,1}^{(4)} &= \frac{11}{6} & c_{1,2}^{(4)} &= \frac{7}{3} & c_{1,3}^{(4)} &= \frac{11}{6} \\
  c_{2,1}^{(4)} &= 1 & c_{2,2}^{(4)} &= 0 & c_{2,3}^{(4)} &= -1 \\
  c_{3,1}^{(4)} &= \frac{1}{6} & c_{3,2}^{(4)} &= -\frac{2}{6} & c_{3,3}^{(4)} &= \frac{1}{6} \\
  c_{1,1}^{(5)} &= \frac{25}{12} & c_{1,2}^{(5)} &= \frac{35}{12} & c_{1,3}^{(5)} &= \frac{35}{12} & c_{1,4}^{(5)} &= \frac{25}{12} \\
  c_{2,1}^{(5)} &= \frac{35}{24} & c_{2,2}^{(5)} &= \frac{5}{8} & c_{2,3}^{(5)} &= -\frac{5}{8} & c_{2,4}^{(5)} &= -\frac{35}{24} \\
  c_{3,1}^{(5)} &= \frac{5}{12} & c_{3,2}^{(5)} &= -\frac{5}{12} & c_{3,3}^{(5)} &= -\frac{5}{12} & c_{3,4}^{(5)} &= \frac{5}{12} \\
  c_{4,1}^{(5)} &= \frac{1}{24} & c_{4,2}^{(5)} &= -\frac{3}{24} & c_{4,3}^{(5)} &= \frac{3}{24} & c_{4,4}^{(5)} &= -\frac{1}{24}
\end{align*}
\]
More explicit coefficients

For brevity, we give the matrices \([(n - 1)!c_{i,j}^{(n)}]\) for \(n = 6, 7\).

\[
\begin{bmatrix}
274 & 404 & 444 & 404 & 274 \\
225 & 150 & 0 & -150 & -225 \\
85 & -40 & -90 & -40 & 85 \\
15 & -30 & 0 & 30 & -15 \\
1 & -4 & 6 & -4 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1764 & 2688 & 3108 & 3108 & 2688 & 1764 \\
1624 & 1330 & 490 & -490 & -1330 & -1624 \\
735 & -105 & -630 & -630 & -105 & 735 \\
175 & -245 & -140 & 140 & 245 & -175 \\
21 & -63 & 42 & 42 & -63 & 21 \\
1 & -5 & 10 & -10 & 5 & -1
\end{bmatrix}
\]

Note that all row sums except the first are 0.
Stirling numbers of the first kind

The first columns the of the matrices above turn out to be well-known. Let $\begin{bmatrix} n \\ k \end{bmatrix}$ be the number of permutations of \{1, 2, \ldots, n\} with exactly $k$ disjoint cycles; this is the unsigned Stirling number of the first kind. We set $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$, and for $n \geq 1$ the Stirling number $\begin{bmatrix} n \\ k \end{bmatrix}$ is only nonzero if $1 \leq k \leq n$. Some useful facts are

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = (n - 1)!, \quad \begin{bmatrix} n \\ n \end{bmatrix} = 1, \quad \begin{bmatrix} n \\ n - 1 \end{bmatrix} = \binom{n}{2}.$$

One has the generating function

$$x(x + 1) \cdots (x + n - 1) = \sum_{k=1}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k, \quad n \geq 1. \quad (6)$$
The following result relates the Stirling numbers to the $c_{k,1}^{(n)}$.

**Lemma (4)**

For $1 \leq k \leq n - 1$,

$$c_{k,1}^{(n)} = (-1)^{k-1} c_{k,n-1} = \frac{1}{(n-1)!} \left[ \begin{array}{c} n \\ k + 1 \end{array} \right].$$

**Proof.**

By Lemma (3), it suffices to prove the last equality. Lemma (2) says

$$c_{k,n-1}^{(n)} = \sum_{1 \leq i_0 < \cdots < i_{k-1} < n} \frac{(-1)^{k-1}}{(n - i_0) \cdots (n - i_{k-1})}.$$
Proof cont’d.

so it’s enough to show

\[\sum_{1 \leq i_0 < \cdots < i_{k-1} < n} \frac{(n-1)!}{(n-i_0) \cdots (n-i_{k-1})} = \binom{n}{k+1}.\]

Now the left-hand side is evidently the sum of all products of \(n-k-1\) distinct factors from the set \(\{1, 2, \ldots, n-1\}\), and that this is \(\binom{n}{k+1}\) follows from consideration of the generating function (6).
A general formula

The preceding result generalizes as follows.

**Lemma (5)**

For $1 \leq k, j \leq n - 1$,

$$
c_{k,j}^{(n)} = \sum_{q=1}^{j} \sum_{p=1}^{q} \frac{(-1)^{p-1} \left\lfloor \frac{q}{p} \right\rfloor \left[ \frac{n+1-q}{k+2-p} \right]}{(q - 1)!(n - q)!}.
$$

We only sketch the proof. The lemma amounts to

$$
c_{k,j}^{(n)} - c_{k,j-1}^{(n)} = \sum_{p=1}^{j} \frac{(-1)^{p-1} \left\lfloor \frac{j}{p} \right\rfloor \left[ \frac{n+1-j}{k+2-p} \right]}{(q - 1)!(n - q)!}.
$$

(7)

By Lemma (3), we can make this about $c_{k,n-j}^{(n)} - c_{k,n-j+1}^{(n)}$. 
Now think of the $c_{k,i}^{(n)}$ as sums of terms via Lemma (2). If $j \leq k$, then all terms contributing to $c_{k,n-j+1}^{(n)}$ also contribute to $c_{k,n-j}^{(n)}$, and in addition there are $j$ classes of terms that contribute to $c_{k,n-j}^{(n)}$ and not to $c_{k,n-j+1}^{(n)}$: these $j$ classes of terms can be matched up with the $j$ terms on the right-hand side of equation (7). If $j > k$, then there are also terms that contribute to $c_{k,n-j+1}^{(n)}$ and not to $c_{k,n-j}^{(n)}$: one can use Lemmas (2) and (4) to match up their sum with a term on the right-hand side of equation (7).
Proof of the second sum formula

Now we’ll prove the sum formulas (1) and (4). As a warm-up, we’ll do the second of these first. Replacing $k$ by $n$, equation (4) is

$$\sum_{a_1 + \cdots + a_n = m, \ a_i \geq 1} H(a_1, \ldots, a_n) = \frac{1}{(n-1)!} \sum_{j=1}^{n-1} \left( \frac{n-2}{j-1} \frac{(-1)^{j-1}}{j^{m-n+1}} \right).$$

The left-hand side is $C(n-1, n; m)$, so it suffices to show that

$$c_{n-1,j}^{(n)} = \frac{(-1)^{j-1}}{(n-1)!} \binom{n-2}{j-1}.$$
Proof of the second sum formula cont’d

Since \( \binom{n}{k} = 0 \) when \( k > n \) and \( \binom{n}{n} = 1 \), this turns out to be an easy consequence of Lemma (5):

\[
C_{n-1,j}^{(n)} = \sum_{q=1}^{j} \sum_{p=1}^{q} \frac{(-1)^{p-1} \binom{q}{p} \binom{n+1-q}{n+1-p}}{(q-1)!(n-q)!}
\]

\[
= \sum_{q=1}^{j} \frac{(-1)^{q-1} \binom{q}{q} \binom{n+1-q}{n+1-q}}{(q-1)!(n-q)!}
\]

\[
= \frac{1}{(n-1)!} \sum_{q=1}^{j}(-1)^{q-1} \binom{n-1}{q-1}
\]

\[
= \frac{(-1)^{j-1}}{(n-1)!} \binom{n-2}{j-1}.
\]
Proof of the first sum formula

Next we attack equation (1). We have

$$\sum_{a_1 + \ldots + a_n = m} H(a_1, \ldots, a_n) = C(0, n; m) + \sum_{k=1}^{n-1} C(k, n; m).$$

Now $C(0, n; m)$ is

$$H(m, 0, \ldots, 0) + H(0, m, 0, \ldots, 0) + \cdots + H(0, \ldots, 0, m)$$

$$= \zeta(m) + \zeta(m) - 1 + \cdots + \zeta(m) - 1 - \frac{1}{2^m} - \cdots - \frac{1}{(n-1)^m}$$

$$= n\zeta(m) - (n-1) - \frac{n-2}{2^m} - \cdots - \frac{1}{(n-1)^m}$$

$$= n\zeta(m) - \sum_{j=1}^{n-1} \frac{n-j}{j^m}.$$
Proof of the first sum formula cont’d

so (using equation (5))

$$
\sum_{a_1+\cdots+a_n=m} H(a_1, \ldots, a_n) = n \zeta(m) - \sum_{j=1}^{n-1} \frac{n-j}{j^m} + \sum_{k=1}^{n-1} \sum_{j=1}^{n-1} \frac{c_{k,j}^{(n)}}{j^{m-k}}.
$$

Hence, to prove the result requires

$$
\sum_{k=1}^{n-1} j^k c_{k,j}^{(n)} = n - j
$$

for $1 \leq j \leq n - 1$. In view of Lemma (5), this is

$$
\sum_{k=1}^{n-1} \sum_{q=1}^{j} \sum_{p=1}^{q} j^k \left(\frac{(-1)^{p-1} [q] [n+1-q]}{[p] [k+2-p]}\right) \frac{(q-1)!(n-q)!}{(q-1)!(n-q)!} = n - j.
$$

(8)
Proof of the first sum formula cont’d

The left-hand side of equation (8) is

$$\sum_{q=1}^{j} \sum_{p=1}^{q} \frac{(-1)^{p-1} [q]_{p}}{(q-1)!(n-q)!} \sum_{k=1}^{n-1} \left[ \begin{array}{c} n + 1 - q \\ k + 2 - p \end{array} \right] j^k.$$

If \( p = 1 \), the inner sum is

$$\sum_{k=1}^{n-1} \left[ \begin{array}{c} n + 1 - q \\ k + 1 \end{array} \right] j^k = \sum_{k=1}^{n} \left[ \begin{array}{c} n + 1 - q \\ k \end{array} \right] j^{k-1} - \left[ \begin{array}{c} n + 1 - q \\ 1 \end{array} \right] =$$

$$\frac{(n+j-q)!}{j(j-1)!} \left[ \begin{array}{c} n + 1 - q \\ 1 \end{array} \right] = (n-q)! \left[ \begin{array}{c} n+j-q \\ j \end{array} \right] - 1,$$

where we have used the generating function (6).
Proof of the first sum formula cont’d

If $p \geq 2$, the inner sum is

$$\sum_{k \geq 1} \left[ n + 1 - q \right] j^{k+p-2} = j^{p-2} \frac{(n+j-q)!}{(j-1)!},$$

again using equation (6), and so

$$\sum_{k=1}^{n-1} \left[ n + 1 - q \right] j^k = (n-q)! j^{p-1} \binom{n+j-q}{j}.$$

Thus the left-hand side of equation (8) is

$$\sum_{q=1}^{j} \binom{n+j-q}{j} - j + \sum_{q=2}^{j} \sum_{p=2}^{q} \frac{(-1)^{p-1} \left[ q \right]}{(q-1)!} \binom{n+j-q}{j} j^{p-1}.$$
Proof of the first sum formula cont’d

The double sum can be rewritten as

\[
\sum_{q=2}^{j} \binom{n+j-q}{j} \frac{1}{(q-1)!} \sum_{p=2}^{q} \left[ \frac{q}{p} \right] (-j)^{p-1} =
\]

\[
\sum_{q=2}^{j} \binom{n+j-q}{j} \frac{1}{(q-1)!} \left[ (1-j)(2-j) \cdots (q-1-j) - \left[ \frac{q}{1} \right] \right]
\]

\[
= - \sum_{q=2}^{j} \binom{n+j-q}{j} + \sum_{q=2}^{j} \binom{n+j-q}{j} \frac{j-1}{q-1} (-1)^{q-1}
\]

using equation (6) again, so the left-hand side of equation (8) is

\[-j + \sum_{q=1}^{j} \binom{n+j-q}{j} \frac{j-1}{q-1} (-1)^{q-1}.\]
One more lemma

This means we need one last lemma.

**Lemma**

For positive integers $n$ and $j$,

$$
\sum_{q=1}^{j} \binom{n + j - q}{j} \binom{j - 1}{q - 1} (-1)^{q-1} = n.
$$

**Proof.**

Use Wilf’s “snake oil method”: let

$$
F(x) = \sum_{n=1}^{\infty} \sum_{q=1}^{j} \binom{n + j - q}{j} \binom{j - 1}{q - 1} (-1)^{q-1} x^n.
$$
Generalized Harmonic Series (NOT MZVs)

ME Hoffman

Outline
Introduction
\(H\)-series and Stirling numbers of the first kind

Proof of the sum formulas

One more lemma cont’d

Proof cont’d.

Then

\[
F(x) = \sum_{q=1}^{j} \frac{(j - 1)}{(q - 1)}(-1)^{q-1} \sum_{n=1}^{\infty} \binom{n + j - q}{j} x^n
= \sum_{q=1}^{j} \frac{(j - 1)}{(q - 1)} \frac{(-1)^{q-1} x^q}{(1 - x)j+1}
= \frac{x}{(1 - x)j+1} \sum_{p=0}^{j-1} \binom{j - 1}{p} (-x)^p
= \frac{x(1 - x)^{j-1}}{(1 - x)^{j+1}} = \frac{x}{(1 - x)^2} = x + 2x^2 + 3x^3 + \cdots
\]

and the result follows.