Multiple Zeta Values of Even Arguments (and Some Analogous Results)

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USNA Mathematics Colloquium
12 November 2014
Multiple Zeta Values

The multiple zeta values (MZVs) are defined by

\[ \zeta(i_1, \ldots, i_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}}, \]

where \( i_1, \ldots, i_k \) are positive integers with \( i_1 > 1 \). We call \( k \) the depth of the MZV, and \( i_1 + \cdots + i_k \) its weight. There are many remarkable relations among MZVs, starting with \( \zeta(2, 1) = \zeta(3) \) (which is often rediscovered and posed as a problem, but was known to Euler). One interesting fact is that all known relations are homogeneous by weight, but we are very far from proving this must be true.
Euler studied MZVs of depth 1 (which are just the values of the “Riemann” zeta function) and obtained the famous formula

\[ \zeta(2n) = \frac{(-1)^{n-1} B_{2n} (2\pi)^{2n}}{2(2n)!}. \]

Somewhat less famously, he also studied MZVs of depth 2, and of the many formulas he gave two are of interest here. First, he found that the sum of all the depth-2 MZVs of weight \( m \) is just \( \zeta(m) \), that is

\[ \sum_{i=2}^{m-1} \zeta(i, m - i) = \zeta(m). \]

Second, he gave an alternating sum formula for even weights:

\[ \sum_{i=2}^{2n-1} (-1)^i \zeta(i, 2n - i) = \frac{1}{2} \zeta(2n). \]
Recent History

When MZVs of arbitrary depth started to be studied around 1988, one of the first results to attract interest was the generalization of Euler’s sum theorem

\[ \sum_{i_1,\ldots,i_k=m, \ i_1>1} \zeta(i_1, \ldots, i_k) = \zeta(m). \]

This was conjectured by C. Moen (who proved it for \( k = 3 \)) and proved in general by A. Granville and D. Zagier. More recently there has been interest in the sums of MZVs of fixed weight with even arguments, i.e.,

\[ E(2n, k) = \sum_{i_1+\cdots+i_k=n} \zeta(2i_1, \ldots, 2i_k). \]
Formulas for Even-Argument Sums

If you add Euler’s sum theorem in weight $2n$ to his alternating sum theorem in the same weight, you get

$$E(2n, 2) = \frac{3}{4}\zeta(2n).$$

(Though implicit in Euler, this result seems to have been first stated in print by H. Gangl, M. Kaneko and D. Zagier in 2006.) Now it is true that $E(2n, k)$ is always a rational multiple of $\zeta(2n)$ (as I will shortly explain), but for $k > 2$ this multiple depends on the weight $2n$. Nevertheless, it is possible to get formulas with weight-free coefficients if extra terms are allowed.
The Shen-Cai Formulas

In 2012 Z. Shen and T. Cai published the following formulas for even-argument sums of depth 3 and 4:

\[ E(2n, 3) = \frac{5}{8}\zeta(2n) - \frac{1}{4}\zeta(2)\zeta(2n - 2) \]
\[ E(2n, 4) = \frac{35}{64}\zeta(2n) - \frac{5}{16}\zeta(2)\zeta(2n - 2). \]

What caught my eye was the coefficient of \( \zeta(2n) \). Along with \( E(2n, 1) = \zeta(2n) \) and \( E(2n, 2) = \frac{3}{4}\zeta(2n) \) we have a sequence

\[ 1, \quad \frac{3}{4}, \quad \frac{5}{8}, \quad \frac{35}{64}, \quad \ldots \]
The Shen-Cai Formulas

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\[ 1, \quad \frac{3}{4}, \quad \frac{5}{8}, \quad \frac{35}{64}, \quad \ldots \]

\[ 1, \quad 1 \cdot \frac{3}{4}, \quad 1 \cdot \frac{3}{4} \cdot \frac{5}{6}, \quad 1 \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8}, \quad \ldots \]
Once I saw the formula for the leading coefficient, I was hooked. I wouldn’t be satisfied until I had a general formula for \( E(2n, k) \). In fact I worked the whole thing out in the ten days from 21 May 2012 (when I started working in earnest) to 31 May 2012 (when I submitted a preprint to the arXiv). It’s the fastest I’ve ever written a paper, and it was the excellent working environment at DESY Zeuthen that enabled me to concentrate so fully on a single problem. I thank DESY Zeuthen (and particularly Johannes Blümlein) for providing the environment, and the DAAD (German Academic Exchange Service) for providing the funding that made my visit possible. Vielen Dank!
Symmetric Sums

Now I’ll explain why $E(2n, k) \in \mathbb{Q}\pi^{2n}$.

Theorem (Hoffman, 1992)

Let $i_1, \ldots, i_k$ be integers larger than 1. If $I = (i_1, \ldots, i_k)$ and the symmetric group $S_k$ acts on $I$ in the obvious way, then

$$\sum_{\sigma \in S_k} \zeta(\sigma \cdot I) = \sum_{\text{partitions } \Pi \text{ of } \{1, \ldots, k\}} c(\Pi) \zeta(I, \Pi),$$

where for the partition $\Pi = \{P_1, \ldots, P_t\}$ we define

$$c(\Pi) = (-1)^{k-t}(|P_1| - 1)! \cdots (|P_t| - 1)!$$

and

$$\zeta(I, \Pi) = \prod_{s=1}^{t} \zeta \left( \sum_{j \in P_s} i_j \right).$$
This theorem shows that any symmetric sum of MZVs (i.e., invariant under permutation of the exponent string) is a rational linear combination of products of ordinary MZVs. Thus $\zeta(6, 2) + \zeta(2, 6) = \zeta(6)\zeta(2) - \zeta(8)$, but it is unknown if $\zeta(6, 2)$ is such a linear combination. In my 1992 paper I used the theorem to prove that

$$E(2n, n) = \zeta\left(\underbrace{2, 2, \ldots, 2}_n\right) = \frac{\pi^{2n}}{(2n + 1)!}.$$

Clearly any $E(2n, k)$ is a symmetric sum, and although the details look messy it must be linear combination of products of even zetas homogeneous by weight, and so a rational multiple of $\pi^{2n}$ (hence of $\zeta(2n)$).
But *Which* Rational Multiple?

The problem with using my theorem to actually compute things is that it’s a sum over set partitions, which increase according to Bell numbers

\[
1, 2, 5, 15, 52, 203, 877, 4140 \ldots
\]

Further, the \( E(2n, k) \) get to be more complicated symmetric sums as \( k \) gets smaller than \( n \):

\[
E(2n, n-1) = \zeta(4,2,\ldots,2) + \zeta(2,4,2,\ldots,2) + \cdots + \zeta(2,\ldots,2,4)
\]

\[
E(2n, n-2) = \zeta(4,4,2,\ldots,2) + \cdots + \zeta(2,2,\ldots,2,4,4) + \zeta(6,2,\ldots,2) + \cdots + \zeta(2,\ldots,2,6)
\]
For example, it takes a fairly heroic calculation to get $E(14, 5)$ directly from my theorem: one has to add two sums of 52 products each, which eventually reduces to

\[
\frac{1}{24} \zeta(2)^4 \zeta(6) + \frac{1}{12} \zeta(2)^3 \zeta(4)^2 - \frac{3}{4} \zeta(2)^2 \zeta(4) \zeta(6) - \frac{1}{4} \zeta(2)^3 \zeta(8)
\]
\[
- \frac{1}{4} \zeta(2) \zeta(4)^3 + \frac{7}{4} \zeta(2) \zeta(4) \zeta(8) + \frac{19}{24} \zeta(4)^2 \zeta(6) + \frac{5}{6} \zeta(2) \zeta(6)^2
\]
\[
+ \zeta(2)^2 \zeta(10) - \frac{7}{4} \zeta(6) \zeta(8) - 2 \zeta(4) \zeta(10) - \frac{5}{2} \zeta(2) \zeta(12) + 3 \zeta(14)
\]
\[
= \frac{\pi^{14}}{29189160000}.
\]

Fortunately I didn’t persist in this foolishness for too long.
But *Which* Rational Multiple? cont’d

By May 23rd I could prove that

\[ E(2k, k) = \frac{\pi^{2k}}{(2k + 1)!} \]

has some only slightly more complicated cousins, e.g.,

\[ E(2k + 2, k) = \frac{k\pi^{2k+2}}{3(2k + 1)!(2k + 3)} \]
\[ E(2k + 4, k) = \frac{k(7k + 13)\pi^{2k+4}}{90(2k + 1)!(2k + 3)(2k + 5)} , \]

the latter of which makes the calculation of \( E(14, 5) \) *much* faster. I’ll say where these formulas came from later, but first I’ll explain what I did with them.
Guessing the General Theorem

My goal was to come up with extensions of the Shen-Cai formulas. Following the pattern of the leading coefficients, I decided the formula for $k = 5$ should look like

$$E(2n, 5) = \frac{63}{128} \zeta(2n) + a\zeta(2)\zeta(n - 2) + b\zeta(4)\zeta(n - 4)$$

for constants $a$ and $b$. Setting $n = 5$ and $n = 6$ gave me two equations in two unknowns, which I solved to get $a = -\frac{21}{64}$ and $b = \frac{3}{64}$. So my guess was now

$$E(2n, 5) = \frac{63}{128} \zeta(2n) - \frac{21}{64} \zeta(2)\zeta(n - 2) + \frac{3}{64} \zeta(4)\zeta(n - 4),$$

which I tested at $n = 7$: this is why I wanted to know $E(14, 5)$. The formula passed the test, so I worked out a conjectural formula for $E(2n, 6)$ in a similar way.
I now assumed that the general formula looked like

$$E(2n, k) = \frac{1}{2^{2(k-1)}} \binom{2k-1}{k} \zeta(2n) + \sum_{j=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor} c_{k,j} \zeta(2j) \zeta(2n - 2j),$$

and by solving systems with $k$ close to $n$ I got the following coefficients $c_{k,j}$ (starting with $k = 3$)

$k = 3$: $c_{3,1} = -\frac{1}{4}$

$k = 4$: $c_{4,1} = -\frac{5}{16}$

$k = 5$: $c_{5,1} = -\frac{21}{64}$, $c_{5,2} = \frac{3}{64}$

$k = 6$: $c_{6,1} = -\frac{64}{21}$, $c_{6,2} = \frac{21}{256}$

$k = 7$: $c_{7,1} = -\frac{165}{512}$, $c_{7,2} = \frac{27}{256}$, $c_{7,3} = -\frac{3}{1024}$

$k = 8$: $c_{8,1} = -\frac{1287}{4096}$, $c_{8,2} = \frac{495}{4096}$, $c_{8,3} = -\frac{27}{4096}$
Guessing Coefficients

All these coefficients are consistent with the formulas

\[ c_{k,1} = \frac{-1}{2^{2(k-2)}} \binom{2k - 3}{k} \]
\[ c_{k,2} = \frac{3}{2^{2(k-2)}} \binom{2k - 5}{k} \]
\[ c_{k,3} = \frac{-3}{2^{2(k-2)}} \binom{2k - 7}{k} \]

so I decided that

\[ c_{k,j} = \frac{a_j}{2^{2(k-2)}} \binom{2k - 2j - 1}{k} \]

for some sequence \( a_1, a_2, \ldots \).
What is the Next Term of this Sequence?

Now I pressed ahead, just solving for $a_j$ by using the equation $E(2k, k) = \pi^{2k}/(2k + 1)!$ repeatedly. So my sequence was

$$a_1 = -1, \ a_2 = 3, \ a_3 = -3,$$
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$$a_1 = -1, \quad a_2 = 3, \quad a_3 = -3, \quad a_4 = \frac{5}{3}, \quad a_5 = -\frac{3}{5}, \quad a_6 = \frac{105}{691}$$
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$$a_j = \frac{-1}{(4j + 2)B_{2j}}$$

and so

$$c_{k,j} = \frac{-1}{2^{2k-3}(2j + 1)B_{2j}} \binom{2k - 2j - 1}{k}.$$
The Formula at Last

On May 26th I had my formula:

\[ E(2n, k) = \frac{\zeta(2n)}{2^{2k-2}} \binom{2k-1}{k} \]

\[ - \sum_{j=1}^{\lfloor k/2 \rfloor} \frac{\zeta(2j)\zeta(2n-2j)}{2^{2k-3}(2j+1)B_{2j}} \binom{2k-2j-1}{k}. \]

Since analytic continuation gives \( \zeta(0) = -\frac{1}{2} \), this can be written

\[ E(2n, k) = - \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{\zeta(2j)\zeta(2n-2j)}{2^{2k-3}(2j+1)B_{2j}} \binom{2k-2j-1}{k}. \]
Symmetric Functions

Proving this formula (which took me four more days) was a quite different exercise from discovering it. It involved a detour through the algebra Sym of symmetric functions. Here are some definitions.

Let $x_1, x_2, \ldots$ all have degree 1, and let $\mathcal{P} \subset \mathbb{Q}[[x_1, x_2, \ldots ]]$ be the set of formal power series in the $x_i$ of bounded degree.

Then $\mathcal{P}$ is a graded $\mathbb{Q}$-algebra. An element $f \in \mathcal{P}$ is symmetric if the coefficient of any term

$$x_{i_1}^{n_1} \cdots x_{i_k}^{n_k}$$

with $i_1, \ldots, i_k$ distinct, agrees with that of

$$x_1^{n_1} \cdots x_k^{n_k}.$$ 

The set of such $f$ forms an algebra $\text{Sym} \subset \mathcal{P}$. 
For any integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, the monomial symmetric function $m_\lambda$ is the “smallest” symmetric function containing the monomomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_k^{\lambda_k}$. For example,

$$m_{21} = x_1^2 x_2 + x_1^2 x_3 + \cdots + x_2^2 x_3 + \cdots + x_1 x_2^2 + x_1 x_3^2 + \cdots$$

The set $\{ m_\lambda | \lambda \text{ is a partition} \}$ is an integral basis for Sym. The elementary symmetric functions are

$$e_k = m_{(1^k)},$$

where $(1^k)$ is the partition of $k$ into 1’s. The complete symmetric functions are

$$h_k = \sum_{\lambda \text{ is a partition of } k} m_\lambda.$$
The **power-sum** symmetric functions are

\[ p_k = m(k). \]

If for a partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) we let

\[ e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k} \]

and similarly \( h_\lambda \) and \( p_\lambda \), then it is well known that

\[ \{ e_\lambda | \lambda \text{ is a partition} \} \text{ and } \{ h_\lambda | \lambda \text{ is a partition} \} \]

are integral bases for Sym, while

\[ \{ p_\lambda | \lambda \text{ is a partition} \} \]

is a rational basis.
Generating Functions

The generating function of the \textbf{elementary} symmetric functions is

\[ E(t) = \sum_{i \geq 0} e_i t^i = \prod_{i \geq 1} (1 + tx_i), \]

while that for the \textbf{complete} symmetric functions is

\[ H(t) = \sum_{i \geq 0} h_i t^i = \prod_{i \geq 1} \frac{1}{1 - tx_i} = E(-t)^{-1}. \]

The logarithmic derivative of the latter is the generating function \( P(t) \) for the \textbf{power sums}:

\[ P(t) = \frac{d}{dt} \log H(t) = -\sum_{i \geq 1} \frac{d}{dt} \log(1 - tx_i) = \sum_{i \geq 1} p_i t^{i-1}. \]
There is a homomorphism $\mathcal{Z} : \text{Sym} \to \mathbb{R}$ induced by sending $x_i$ to $1/i^2$. Then

$$\mathcal{Z}(p_k) = \zeta(2k), \quad \mathcal{Z}(e_k) = E(2k, k) = \frac{\pi^{2k}}{(2k + 1)!}$$

and

$$\mathcal{Z}(h_k) = \sum_{i=1}^{k} E(2k, i).$$

For $k \leq n$, we define symmetric functions $N_{n,k}$ so that $\mathcal{Z}(N_{n,k}) = E(2n, k)$: that is, $N_{n,k}$ is the sum of all monomial symmetric functions corresponding to partitions of $n$ having length $k$. 
Now one of the first things I proved about the \( N_{n,k} \) was the recurrence

\[
p_1 N_{n-1,k} + p_2 N_{n-2,k} + \cdots + p_{n-k} N_{k,k} = (n-k) N_{n,k} + (k+1) N_{n,k+1},
\]

which somewhat resembles the standard recurrence

\[
p_1 h_{n-1} + p_2 h_{n-2} + \cdots + p_{n-1} h_1 + p_n = nh_n
\]
of power-sum and complete symmetric functions. In particular,

\[
p_1 N_{k,k} = N_{k+1,k} + (k+1) N_{k+1,k+1},
\]
or

\[
N_{k+1,k} = p_1 N_{k,k} - (k+1) N_{k+1,k+1}.
\]
Recurrences cont’d

Apply $3$ to the latter equation to get

$$E(2k + 2, k) = \zeta(2)E(2k, k) - (k + 1)E(2k + 2, k + 1)$$

leading to the formula

$$E(2k + 2, k) = \frac{\pi^{2m+2}}{6(2k + 1)!} - \frac{(k + 1)\pi^{2k+2}}{(2k + 3)!}$$

$$= \frac{k\pi^{2k+2}}{3(2k + 1)!(2k + 3)}$$

we saw earlier; the formula for $E(2k + 4, k)$ arises in a similar way. In fact, the desire for such formulas is what led me to prove the recurrence.
Generating Function for the $N_{n,k}$

It’s actually easy to write the generating function

$$\mathcal{F}(t, s) = 1 + \sum_{n \geq k \geq 1} N_{n,k} t^n s^k,$$

if we remember that the power of $t$ keeps track of degree, while $s$ must keep track of the number of distinct $x_i$’s involved in a symmetric function:

$$\mathcal{F}(t, s) = \prod_{i=1}^{\infty} \frac{1 + (s - 1)tx_i}{1 - tx_i} = H(t)E((s - 1)t).$$
Now let’s apply $\zeta$ to this result: if

$$F(t, s) = 1 + \sum_{n\geq k \geq 1} E(2n, k) t^n s^k$$

is the generating function for the $E(2n, k)$, we have

$$F(t, s) = \zeta(F(t, s)) = \zeta(H(t)) \zeta(E((s - 1)t))$$

From our formula for $\zeta(e^t)$ we have

$$\zeta(E(t)) = \sum_{i=0}^{\infty} \frac{\pi^{2i} t^i}{(2i + 1)!} = \frac{\sinh(\pi \sqrt{t})}{\pi \sqrt{t}}.$$
Generating Function for the $E(2n, k)$ cont’d

But $H(t) = E(-t)^{-1}$, so

$$3(H(t)) = \frac{\pi \sqrt{-t}}{\sinh \pi \sqrt{-t}} = \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}}$$

and thus

$$F(t, s) = 3(H(t))3(E(-(1 - s)t)) = \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}} \frac{\sin(\pi \sqrt{1 - s\sqrt{t}})}{\pi \sqrt{1 - s\sqrt{t}} \sin \pi \sqrt{1 - s\sqrt{t}}}$$

$$= \frac{\sin(\pi \sqrt{t}\sqrt{1 - s})}{\sqrt{1 - s \sin \pi \sqrt{t}}}. $$
Using the Generating Function

Armed with the generating function, one can use any computer algebra system that does series (like PARI or Maple) to spit out as many of the $E(2n, k)$ as desired. This makes it easy to compute not only

$$E(14, 5) = \frac{\pi^{14}}{29189160000}$$

but also, say,

$$E(28, 9) = \frac{19697\pi^{28}}{1423274702804081487360000000}.$$

(In practice one finds that the hardest thing is finding the right term in the output and reading out all the digits correctly.)
Expanding out the Generating Function

There’s still the problem of proving the conjecture. We want a formula for $E(2n, k)$ for fixed $k$, which means we want to look at the coefficient of $s^k$ in $F(t, s)$. Writing

$$F(t, s) = 1 + \sum_{k \geq 1} s^k G_k(t),$$

then it is easy to see from the explicit formula that

$$G_k(t) = (-1)^k \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}} \sum_{j \geq k} \frac{(-1)^j \pi^{2j} t^j}{(2j + 1)!} {j \choose k}$$

$$= (-1)^k \frac{\pi \sqrt{t}}{\sin \pi \sqrt{t}} \frac{(t)^k}{k!} \frac{d^k}{dt^k} \left( \frac{\sin \pi \sqrt{t}}{\pi \sqrt{t}} \right).$$
A Little Calculus Problem

So we have the small calculus problem of finding the $n$th derivative of $\sin(\pi \sqrt{t}) / \pi \sqrt{t}$. Evidently

\[
\frac{(-t)^n}{n!} \frac{d^n}{dt^n} \left( \frac{\sin \pi \sqrt{t}}{\pi \sqrt{t}} \right) = P_n(\pi^2 t) \cos \pi \sqrt{t} + Q_n(\pi^2 t) \frac{\sin \pi \sqrt{t}}{\pi \sqrt{t}}
\]

for some polynomials $P_n, Q_n$. 
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for some polynomials $P_n$, $Q_n$. With a little help from Maple I found that

$$P_n(x) = -\sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{(-4x)^j}{2^{2n-1}(2j+1)!} \binom{2n-2j-1}{n}$$

and

$$Q_n(x) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-4x)^j}{2^{2n}(2j)!} \binom{2n-2j}{n}.$$
Expanding out the Generating Function

Making use of our calculus exercise, we can now write

\[ G_k(t) = -\sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-1}(2j+1)!} \binom{2k-2j-1}{k} \pi \sqrt{t} \cot \pi \sqrt{t} \]

\[ + \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k}(2j)!} \binom{2k-2j}{k} \]

\[ = \frac{1}{2}(1 - \pi \sqrt{t} \cot \pi \sqrt{t}) \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!} \binom{2k-2j-1}{k} \]

\[ + \text{terms of degree } < k. \]
Now going back to the statement of our conjecture, we can write it a bit more simply if we use Euler’s formula for $\zeta(2j)$; this cancels the Bernoulli number in the denominator, leaving us with the cleaner version

\[
E(2n, k) = \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^j \frac{\pi^{2j} \zeta(2n - 2j)}{2^{2k-2j-2}(2j + 1)} \left(\frac{2k - 2j - 1}{k}\right).
\]

Thus, we will be done if we can show $G_k(t)$ is equal to

\[
\sum_{n \geq k} t^k \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^j \frac{\pi^{2j} \zeta(2n - 2j)}{2^{2k-2j-2}(2j + 1)} \left(\frac{2k - 2j - 1}{k}\right).
\]
Restating the Conjecture cont’d

Write the latter sum in the form

\[
\left\lfloor \frac{k-1}{2} \right\rfloor \sum_{j=0} \frac{(-4\pi^2 t)^j}{2^{2k-2}(2j+1)!} \binom{2k - 2j - 1}{k} \sum_{n\geq k} \zeta(2n - 2j) t^{n-j},
\]

and note that

\[
\sum_{n \geq k} \zeta(2n - 2j) t^{n-j} = \sum_{n \geq j+1} \zeta(2n - 2j) t^{n-j} - \sum_{n=j+1} \zeta(2n - 2j) t^{n-j}
\]

\[
= \sum_{m \geq 2} \zeta(m) t^m - \sum_{n=j+1} \zeta(2n - 2j) t^{n-j}
\]

\[
= \frac{1}{2} \left( 1 - \pi \sqrt{t} \cot \pi \sqrt{t} \right) - \sum_{n=j+1} \zeta(2n - 2j) t^{n-j}.
\]
Hence the conjecture reads

\[
G_k(t) = \frac{1}{2} \left( 1 - \pi \sqrt{t} \cot \pi \sqrt{t} \right) \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-4\pi^2 t)^j}{2^{2k-2} (2j + 1)!} \binom{2k - 2j - 1}{k}
\]

+ terms of degree \(< k\),

where the “terms of degree \(\leq k\)” are exactly those necessary to cancel the terms of degree \(\leq k\) in the preceding sum. But this agrees with the result we got by expanding out the generating function, so we’re done.
Another Formula for $E(2n, k)$

We can get another sum for $E(2n, k)$ as follows. Using the recurrence mentioned earlier, we can obtain an explicit formula for $N_{n,k} = \sum_{|\pi|=n, \text{length}(\pi)=k} m_{\pi}$ in terms of elementary and complete symmetric functions:

$$N_{n,k} = \sum_{i=0}^{n-k} \binom{n-i}{k} (-1)^{n-k-i} h_{i} e_{n-i}.$$ 

Apply 3 to get

$$E(2n, k) = \frac{(-1)^{n-k-1} \pi^{2n}}{(2n+1)!} \sum_{i=0}^{n-k} \binom{n-i}{k} \binom{2n+1}{2i} 2(2^{2i-1}-1) B_{2i}.$$ 

Notice that this has $n - k + 1$ terms rather than $\lfloor \frac{k-1}{2} \rfloor + 1$ terms as the other formula does.
Imitation is the Sincerest Form of Flattery

It wasn’t long after I posted my paper on the arXiv that the industrious Jianqing Zhao produced an analogue. (His preprint was posted on July 10.) Define the multiple $t$-value $t(i_1, \ldots, i_k)$ to be the sum of all the terms in $\zeta(i_1, \ldots, i_k)$ with odd denominators. Then $t(n) = (1 - 2^{-n})\zeta(n)$ for $n \geq 2$. We can define

$$T(2n, k) = \sum_{i_1 + \cdots + i_k = n} t(2i_1, \ldots, 2i_k).$$

Zhao showed that

$$T(2n, k) = \sum_{j=0}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \frac{(-1)^j \pi^{2j} t(2n - 2j)}{2^{2k-2}(2j)!k} \binom{2k - 2j - 2}{k - 1}.$$
Zhao deduces his formula from the generating function

\[ 1 + \sum_{n \geq k \geq 1} T(2n, k) t^n s^k = \frac{\cos \left( \frac{\pi}{2} \sqrt{(1 - s)t} \right)}{\cos \left( \frac{\pi}{2} \sqrt{t} \right)}, \quad (1) \]

whose proof is entirely analogous to that for

\[ 1 + \sum_{n \geq k \geq 1} E(2n, d) t^n s^k = \frac{\sin(\pi \sqrt{(1 - s)t})}{\sqrt{1 - s \sin(\pi \sqrt{t})}}. \]

The homomorphism \( \mathcal{T} : \text{Sym} \rightarrow \mathbb{R} \) sending \( x_i \) to \( \frac{1}{(2i-1)^2} \) sends \( p_k \) to \( t(2k) \), from which it is easy to see that

\[ \mathcal{T}(P(t)) = \frac{\pi}{4} \sqrt{t} \tan \left( \frac{\pi}{2} \sqrt{t} \right). \]

From this one finds \( \mathcal{T}(H(t)) \) and \( \mathcal{T}(E(t)) \), and equation (1) follows by applying \( \mathcal{T} \) to \( \mathcal{F}(s, t) = H(t)E((s - 1)t) \).
Another Extension

Not content with the $t$-value extension, Zhao wrote another paper jointly with H. Yuan examining the sum of multiple zeta values whose arguments are multiples of 4. Define

$$Q(4n, k) = \sum_{i_1 + \cdots + i_k = n} \zeta(4i_1, 4i_2, \ldots, 4i_k).$$

The generating function

$$G(t, s) = 1 + \sum_{n \geq k \geq 1} Q(4n, k) t^n s^k$$

is actually not hard to obtain: it can be written

$$G(t, s) = \frac{\sin \left( \pi \sqrt[4]{(1 - s)t} \right) \sinh \left( \pi \sqrt[4]{(1 - s)t} \right)}{\sqrt{1 - s} \sin \left( \pi \sqrt[4]{t} \right) \sinh \left( \pi \sqrt[4]{t} \right)}.$$
The formula for $G(t, s)$ is again obtained by applying an appropriate homomorphism to the identity

$$\mathcal{F}(t, s) = H(t)E((s - 1)t).$$

Here we need $Q : \text{Sym} \to \mathbb{R}$ defined by sending $x_i$ to $\frac{1}{i^4}$. Then it is well known that

$$Q(E(-t)) = \frac{\sinh(\pi \sqrt[4]{t}) \sin(\pi \sqrt[4]{t})}{\pi^2 \sqrt{t}},$$

from which the formula for $G(t, s)$ follows. The hard part of the Yuan-Zhao result is their formula for $Q(4n, k)$, which is not easy to fit on a slide. Their version of the “calculus exercise” required to prove it requires four functions rather than two.
Another Extension cont’d

Nevertheless, the first few cases are rather nice. First,

\[ Q(4n, 2) = \frac{1}{2} \sum_{k=1}^{n-1} \zeta(4k) \zeta(4n - 4k) - \frac{n - 1}{2} \zeta(4n). \]

Using this, one has two formulas reminiscent of those proved by Shen and Cai:

\[ Q(4n, 3) = -\frac{5}{16} \zeta(4n) + \frac{1}{8} \zeta(4n - 2) \zeta(2) + \frac{5}{4} Q(4n, 2) \]

\[ Q(4n, 4) = -\frac{115}{64} \zeta(4n) + \frac{13}{64} \zeta(4n - 2) \zeta(2) + \frac{85}{64} Q(4n, 2). \]
Further Extensions

Further extensions are certainly possible. For example, if we let

\[ S(6n, k) = \sum_{i_1+\ldots+i_k=n} \zeta(6i_1, \ldots, 6i_k), \]

then from known results it is easy to show that

\[ 1 + \sum_{n \geq k \geq 1} S(6n, k) t^n s^k = \]
\[ \sin(\pi^{\frac{6}{\sqrt{t}}}(1 - s))(\cosh(\pi^{\frac{6}{\sqrt{3}}t}(1 - s))) - \cos(\pi^{\frac{6}{\sqrt{t}}}(1 - s))) \]
\[ \sqrt{1 - s} \sin(\pi^{\frac{6}{\sqrt{t}}})(\cosh(\pi^{\frac{6}{\sqrt{3}}t}) - \cos(\pi^{\frac{6}{\sqrt{t}}})) \]

But the complexity of the explicit formula Zhao and Yuan got for \( Q(4n, k) \) suggests that an explicit formula for \( S(6n, k) \) would be a real mess.