# Calculating with Pictures, or How to Stay (Mostly) Sane in a Pandemic

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# Outline

# Introduction



This talk is about multiple zeta values and multiple zeta-star values, about which I've spoken before. First let's get the definitions out of the way. For positive integers $i_1, \dots, i_k$ with $i_1 > 1$, the corresponding multiple zeta value (MZV) is the real number

$$\zeta(i_1, i_2, \dots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

The number $k$ is called the depth, and $i_1 + i_2 + \cdots + i_k$ is called the weight.

# Introduction cont'd



Multiple zeta-star values (MZSVs) are defined similarly:

$$\zeta^\star(i_1, i_2, \dots, i_k) = \sum_{n_1 \ge n_2 \ge \cdots \ge n_k \ge 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

The only difference from the previous definition is the non-strict inequalities in the summation indices. Clearly multiple zeta and multiple zeta-star values are related in a simple way: for example,

$$\zeta^\star(2,1,3) = \zeta(2,1,3) + \zeta(3,3) + \zeta(2,4) + \zeta(6).$$

I will call this the "collapsing sum": if the depth is $d$, there are $2^{d-1}$ terms.

# Introduction cont'd

Multiple zeta values really go back to Euler for depth two, but interest in those of general depth took off in the 1990's with their simultaneous appearance in knot theory and theoretical physics. The first multiple zeta identity is $\zeta(2,1)=\zeta(3)$, i.e,

$$\sum_{n=2}^{\infty}\sum_{m=1}^{n-1}\frac{1}{n^2 m}=\sum_{n=1}^{\infty}\frac{1}{n^3}.$$

This neat little identity is often rediscovered and posed as a problem. It has two remarkable generalizations: the sum theorem and the duality theorem. The sum theorem gives the sum of all multiple zeta values of weight $n$ and a fixed depth as $\zeta(n)$. For example,

$$\zeta(4,1)+\zeta(3,2)+\zeta(2,3)=\zeta(3,1,1)+\zeta(2,2,1)+\zeta(2,1,2)=\zeta(5).$$

# Introduction cont'd



The duality theorem gives a dual $\tau(I)$ for each sequence $I$ so that $\zeta(I) = \zeta(\tau(I))$. The dual of (3) is (2, 1), and the dual of (2, 2, 1) is (3, 2). There are lots of other multiple zeta identities, such as

$$\zeta(\underbrace{2, \dots, 2}_{n}) = \frac{\pi^{2n}}{(2n+1)!} \text{ and } \zeta(\underbrace{3, 1, \dots, 3, 1}_{n}) = \frac{2\pi^{4n}}{(4n+2)!}.$$

(In fact, the identity

$$\zeta(2, 1, \underbrace{2, \dots, 2}_{n}, 3) = \zeta(\underbrace{2, \dots, 2}_{n+3}) + 2\zeta(\underbrace{2, \dots, 2}_{n}, 3, 3),$$

which I conjectured at the end of the 1990's, was recently proved by Minoru Hirose and Nobuo Sato.)

# Introduction cont'd

There is a sum theorem for MZSVs, namely

$$\sum_{|I|=n,\ \ell(I)=k} \zeta^\star(I) = \binom{n-1}{k-1}\zeta(n).$$

But there is no nice analogue for the duality theorem. There are, however, many identities for MZSVs: we mention two that will come up later. First,

$$\zeta^\star(\{2\}_n) = (2 - 2^{2-2n})\zeta(2n), \tag{1}$$

where $\{2\}_n$ means $n$ repetitions of 2. Second,

$$\zeta^\star(\{2\}_n, 1) = 2\zeta(2n+1). \tag{2}$$

# MZVs as iterated integrals



MZVs can be represented by iterated integrals as well as by series. For example,

$$\int_0^1 \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_1}{1-t_1} = \int_0^1 \frac{dt_3}{t_3} \int_0^{t_3} \sum_{i\ge 1} \frac{t_2^i}{i} \frac{dt_2}{1-t_2} = \int_0^1 \sum_{i,j\ge 1} \frac{t_3^{i+j}}{i(i+j)} \frac{dt_3}{t_3} = \sum_{i,j\ge 1} \frac{1}{i(i+j)^2} = \zeta(2,1).$$

# Algebraic notation



We can represent iterated integral computations like the preceding by using the notation

$$x \sim \frac{dt}{t}, \quad y \sim \frac{dt}{1-t}$$

so that the form integrated on the preceding slide is $xy^2$. We can then think of $\zeta$ as the function that evaluates the iterated integral, so $\zeta(2,1) = \zeta(xy^2)$, and more generally

$$\zeta(i_1, \dots, i_k) = \zeta(x^{i_1-1}y \cdots x^{i_k-1}y).$$

The monomials "live" in the noncommutative polynomial ring $\mathbb{Q}\langle x, y\rangle$. The change of variable $t \mapsto 1-t$ corresponds to the antiautomorphism $\tau$ of $\mathbb{Q}\langle x, y\rangle$ that exchanges $x$ and $y$, so e.g., $\zeta(3,2) = \zeta(x^2yxy) = \zeta(\tau(x^2yxy)) = \zeta(xyxy^2) = \zeta(2,2,1)$.

# Algebraic notation cont'd



This proves the duality theorem for MZVs. Actually the iterated integral only converges if the monomial begins in $x$ and ends in $y$. Let $\mathfrak{H}$ be the underlying rational vector space of $\mathbb{Q}\langle x, y\rangle$, $\mathfrak{H}^0$ the subspace generated by monomials starting with $x$ and ending in $y$, together with the empty monomial 1. There is a commutative product on $\mathfrak{H}$, namely the shuffle product $\sqcup\!\sqcup$, and $(\mathfrak{H}^0, \sqcup\!\sqcup)$ is a subalgebra of $(\mathfrak{H}, \sqcup\!\sqcup)$. We have, e.g.,

$$xy \sqcup\!\sqcup\, xy = 2xyxy + 4x^2y^2. \tag{3}$$

In fact shuffle product corresponds, via integration by parts, to the product of iterated integrals, so $\zeta$ becomes a homomorphism from $(\mathfrak{H}^0, \sqcup\!\sqcup)$ to the reals.

# The other product



Thus corresponding to Eq. (3) we have

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1) \tag{4}$$

But we can also multiply MZVs as series, e.g.,

$$\zeta(2)^2 = \zeta(4) + 2\zeta(2,2), \tag{5}$$

corresponding to a multiplication $*$ on $\mathfrak{H}^0$ with $xy * xy = x^3y + 2xyxy$. (This is sometimes called the "stuffle" product.) Since $\zeta(2)^2 = \frac{\pi^4}{36} = \frac{5}{2}\zeta(4)$, this implies that $\zeta(2,2) = \frac{3}{4}\zeta(4)$. Comparing Eq. (5) with Eq. (4) gives $\zeta(4) = 4\zeta(3,1)$, or $\zeta(3,1) = \frac{1}{4}\zeta(4)$.

# The other product cont'd



More generally, one defines $*$ inductively by

$$x^p yw * x^q yv = x^p y(w * x^q yv) + x^q y(x^p yw * v) + x^{p+q+1} y(w * v)$$

for monomials $w, v \in \mathfrak{H}^0$. This gives two commutative products on $\mathfrak{H}^0$, the shuffle $⧢$ and the stuffle $*$, with both $\zeta : (\mathfrak{H}^0, ⧢) \to \mathbb{R}$ and $\zeta : (\mathfrak{H}^0, *) \to \mathbb{R}$ being homomorphisms. The products $⧢$ and $*$ extend to the subspace $\mathfrak{H}^1$ of $\mathfrak{H}$ generated by 1 and all monomials ending in $y$, and it is conjectured that all relations of MZVs can be obtained by comparing these two operations on $\mathfrak{H}^1$. For example,

$$y ⧢ x^2 yxy - y * x^2 yxy = x^2 y^2 xy + xyxyxy + x^2 yxy^2 - x^3 yxy - x^2 yx^2 y$$

giving rise to

$$\zeta(3,1,2) + \zeta(2,2,2) + \zeta(3,2,1) = \zeta(4,2) + \zeta(3,3).$$

# Euler sums in terms of MZVs

But what got me started on the subject of this talk looks like something completely different. Back in the 1990's people got excited by formulas like

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k^2} = \frac{17}{4}\zeta(4),$$

where $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$. Nowadays this sort of thing is utterly routine: the left-hand side is

$$\sum_{k=1}^{\infty} \frac{(1+\cdots+\frac{1}{k})^2}{k^2} = \sum_{1\le i\le k} \frac{1}{i^2k^2} + 2\sum_{1\le i<j\le k} \frac{1}{ijk^2}$$

$$= \zeta^\star(2,2) + 2\zeta(2,1,1) + 2\zeta(3,1) = \left(\frac{7}{4} + 2 + \frac{1}{2}\right)\zeta(4)$$

# Euler sums in terms of MZVs cont'd



using the multiple-zeta identities

$$\zeta(2,2)=\frac{3}{4}\zeta(4),\ \zeta(3,1)=\frac{1}{4}\zeta(4),\ \zeta(2,1,1)=\zeta(4).$$

With a little more work and some more identities you can prove, for example,

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^3}=\frac{93}{16}\zeta(6)-\frac{5}{2}\zeta(3)^2. \tag{6}$$

But the calculations are a bit tedious, and there are a surprising number of incorrect identities in the literature. Seeking to tidy things up, I noticed that such identities are often better expressed in terms of multiple zeta-star values rather than multiple zeta values.

# Euler sums in terms of MZSVs

In particular, if $p \geq 2$ it turns out that

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n^p} = 2\zeta^\star(p,1,1) - \zeta^\star(p,2)$$

(so that $\sum_{n\geq 1} \frac{H_n^2}{n^2} = 2\zeta^\star(2,1,1) - \zeta^\star(2,2) = \left(6 - \frac{7}{4}\right)\zeta(4)$), and more generally

$$\sum_{n=1}^{\infty} \frac{H_n^k}{n^p} = \sum_{|I|=k} (-1)^{k-\ell(I)} \binom{k}{I} \zeta^\star(p,I) \qquad (7)$$

where the sum is over all compositions of $k$, $\ell(I)$ is the number of parts of $I$, and $\binom{k}{I}$ is the multinomial coefficient.

# But what is $\zeta^\star(I)$?



But you might question the utility of such formulas: using Eq. (7) with $p = k = 3$ gives

$$\sum_{n=1}^{\infty} \frac{H_n^3}{n^3} = 6\zeta^\star(3,1,1,1) - 3\zeta^\star(3,2,1) - 3\zeta^\star(3,1,2) + \zeta^\star(3,3),$$

which is not obviously equivalent to Eq. (6) above. One needs a table of the $\zeta^\star(I)$ for compositions $I$, which for weight 6 reads in part

$$\zeta^\star(3,3) = \frac{1}{2}\zeta(6) - \frac{1}{2}\zeta(3)^2$$

$$\zeta^\star(3,1,2) = \frac{11}{8}\zeta(6), \quad \zeta^\star(3,2,1) = -\frac{143}{48}\zeta(6) + 3\zeta(3)^2$$

$$\zeta^\star(3,1,1,1) = \frac{1}{12}\zeta(6) + \zeta(3)^2$$

# But what is $\zeta^\star(I)$? cont'd



and substituting these formulas into the equation on the preceding slide gives us Eq. (6). But what, in general, do we write the MZSVs in terms of? The obvious choice is products of the ordinary zeta values $\zeta(i)$, but this seems to break down in weight 8; $\zeta^\star(6,2)$ has resisted any attempt to write it as a rational polynomial in the ordinary zeta values. As it happens, all weight-9 MZVs and MZSVs are rational polynomials in the $\zeta(i)$, but in every weight 10 and above this doesn't seem to be the case.

# MZV tables

We know such things since systematic tables of MZVs as rational polynomials in the $\zeta(i)$ (and certain additional quantities in higher weights) have been computed from known relations of MZVs. This was done through weight 12 by the Lille group (Bigotte et. al.) in the 1990's, and more recently through weight 22 by Blümlein et. al., the latter being known as the "Multiple Zeta Value Data Mine." (By the way, there are $2^{n-2}$ MZVs of weight $n$, while the conjectural dimension of the space of weight-$n$ MZVs is the $n$th Padovan number $P_n$, defined by $P_1 = 0$, $P_2 = P_3 = 1$, and $P_n = P_{n-2} + P_{n-3}$. So a weight 10 table gives $2^8 = 256$ quantities in terms of $P_{10} = 7$ generators, while a weight 22 table gives $2^{20} = 1{,}048{,}576$ quantities in terms of $P_{22} = 200$ generators.)

# Patterns from tables





But there didn't seem to be tables of MZSVs. So I compiled such tables from existing MZV tables, through weight 9. An immediate consequence was that I noticed the general pattern

$$\zeta^\star(I) = (-1)^{\ell(I)-1}\zeta(\bar{I}) \text{ mod decomposables} \tag{8}$$

for any composition $I$, where $\ell(I)$ is the length of $I$ and $\bar{I}$ is the reverse of $I$. I eventually proved this result, in fact in a more general form, as I discussed a couple of years ago. But today I want to talk about extending the tables of MZSVs, using an idea of Shuji Yamamoto.

# Yamamoto's formalism



Here is Yamamoto's idea. He considers iterated integrals over some subset of $[0,1]^n$, involving the two forms

$$\omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1-t}$$

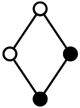
that we called $x$ and $y$ earlier. Let $(X, \delta)$ be a 2-labeled poset, i.e., a finite partially ordered set $X$ with a function $\delta : X \to \{0,1\}$. Call $(X, \delta)$ admissible if $\delta(x) = 1$ for all minimal $x \in X$ and $\delta(x) = 0$ for all maximal $x \in X$. For an admissible 2-labeled poset $(X, \delta)$, define the associated integral by

$$\mathcal{I}(X) = \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta(x)}(t_x), \tag{9}$$

where $\Delta(X) = \{(t_x)_{x\in X} \in [0,1]^X \mid t_x < t_y \text{ if } x < y \text{ in } X\}$.

# Graphical representation

We can represent a 2-labeled poset $(X, \delta)$ graphically by its Hasse diagram, with an open dot $\circ$ for those elements $x \in X$ with $\delta(x) = 0$, and a closed dot $\bullet$ for those $x \in X$ with $\delta(x) = 1$. For example, the 2-labeled poset $X = \{x_1, x_2, x_3, x_4\}$ with $x_1 > x_2 > x_4$, $x_1 > x_3 > x_4$, $\delta(x_1) = \delta(x_2) = 0$, and $\delta(x_3) = \delta(x_4) = 1$ has graphical representation



and associated integral

$$\mathcal{I}(X) = \int_{\substack{t_1 > t_2 > t_4 \\ t_1 > t_3 > t_4}} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{dt_3}{1 - t_3} \frac{dt_4}{1 - t_4}.$$

# 2-labeled posets and MZVs

Admissibility of $(X, \delta)$ guarantees convergence of the integral $\mathcal{I}(X)$. If $X$ is a chain, then $\mathcal{I}(X)$ is just the iterated integral representation of a multiple zeta value described above, e.g.

$\mathcal{I}\Bigg($  $\Bigg) = \displaystyle\int_{t_1<t_2<t_3<t_4<t_5<t_6} \frac{dt_6}{t_6}\frac{dt_5}{t_5}\frac{dt_4}{1-t_4}\frac{dt_3}{1-t_3}\frac{dt_2}{t_2}\frac{dt_1}{1-t_1} =$

$$\int_0^1 \frac{dt_6}{t_6}\int_0^{t_6}\frac{dt_5}{t_5}\int_0^{t_5}\frac{dt_4}{1-t_4}\int_0^{t_4}\frac{dt_3}{1-t_3}\int_0^{t_3}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}$$
$$= \zeta(x^2y^2xy) = \zeta(3,1,2).$$

To get the algebraic description just read the chain from top to bottom, with $x$ for open dots and $y$ for closed ones.

# Properties of the integral



Yamamoto proved the following theorem, which follows easily from general properties of iterated integrals.

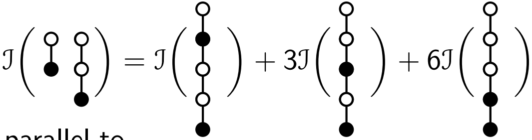
**Theorem**

*Let $X$ be an admissible 2-labeled poset.*

1. *If $Y$ is another admissible 2-labeled poset, $\mathcal{I}(X \amalg Y) = \mathcal{I}(X)\mathcal{I}(Y)$, where $\amalg$ is disjoint union.*
2. *If $a, b \in X$ are incomparable, let $X_{a<b}$ be $X$ with the additional relation $a < b$. Then $\mathcal{I}(X) = \mathcal{I}(X_{a<b}) + \mathcal{I}(X_{b<a})$.*
3. *If $X^\vee$ is $X$ with reversed order and new labeling function $\delta^\vee(x) = 1 - \delta(x)$, then $\mathcal{I}(X^\vee) = \mathcal{I}(X)$.*

# Property 2 and shuffle product

It is evident that combining two disjoint chains via Property 2 is equivalent to shuffle product in $\mathfrak{H}$. For example,



is exactly parallel to

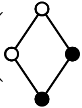
$$xy \sqcup\!\sqcup\, x^2y = xyx^2y + 3x^2yxy + 6x^3y^2,$$

both showing that

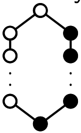
$$\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1).$$

# Using Property 2

In fact, use of Property 2 allows us to write $\mathcal{I}(X)$ as a sum of multiple zeta values for any admissible 2-labeled poset $X$. For example,

$\mathcal{I}\Bigg($  $\Bigg) = \mathcal{I}\Bigg($  $\Bigg) + \mathcal{I}\Bigg($  $\Bigg) = \zeta(3,1) + \zeta(2,2).$

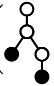
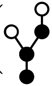
By the way, the sum theorem for MZVs implies that the latter sum is $\zeta(4)$, and more generally that

$\mathcal{I}\Bigg($  $\Bigg) = \zeta(n),$

where the diagram has $n-k$ open dots and $k$ closed ones, for $2 \le k \le n-2$.

# Property 3

Property 3 follows from the change of variable $t \mapsto 1 - t$ in the iterated integral and generalizes the duality of MZVs. It says the value of $\mathcal{I}$ on a labeled poset is unchanged if the poset is "flipped over" and the labels reversed, e.g.,

$\mathcal{I}\Bigg($  $\Bigg) = \mathcal{I}\Bigg($  $\Bigg).$

We note that the weight of a given diagram is simply the number of dots, and the depth is the number of closed dots. All Property 2 transformations preserve weight and depth, while Property 3 takes a diagram of weight $n$ and depth $d$ to a diagram of weight $n$ and depth $n - d$.

# Multiple zeta-star values

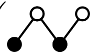
The multiple zeta-star values are easily written in Yamamoto's formalism; in fact

$$\zeta^\star(k_1,\dots,k_r)=\mathcal{I}(X),$$

where $X$ is the poset

$$\{\bar{x}_1<x_2<\cdots<x_{k_1}>\bar{x}_{k_1+1}<x_{k_1+2}<\cdots<x_{k_1+k_2}>\cdots \\ >\bar{x}_{k_1+\cdots+k_{r-1}+1}<x_{k_1+\cdots+k_{r-1}+2}<\cdots<x_{k_1+\cdots+k_r}\},$$

with barred elements having label 1 and all others having label 0. For example,

$\mathcal{I}\Big($  $\Big)$ $$=\int_{t_1<t_2>t_3<t_4}\frac{dt_4}{t_4}\frac{dt_3}{1-t_3}\frac{dt_2}{t_2}\frac{dt_1}{1-t_1}$$

# Multiple zeta-star values cont'd



$$= \int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1-t_3} \int_{t_3}^1 \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_1}{1-t_1}$$

$$= \int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1-t_3} \int_{t_3}^1 \sum_{i\ge 1} \frac{t_2^{i-1}}{i} dt_2 =$$

$$\int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1-t_3} \sum_{i\ge 1} \frac{1-t_3^i}{i^2} = \int_0^1 \sum_{i,j\ge 1} \left[ \frac{t_4^{j-1}}{i^2 j} - \frac{t_4^{i+j-1}}{i^2(i+j)} \right] dt_4$$

$$= \sum_{i=1}^{\infty} \frac{1}{i^2} \sum_{j=1}^{\infty} \left[ \frac{1}{j^2} - \frac{1}{(i+j)^2} \right] = \sum_{i=1}^{\infty} \frac{1}{i^2} \sum_{j=1}^{i} \frac{1}{j^2} = \zeta^\star(2,2).$$

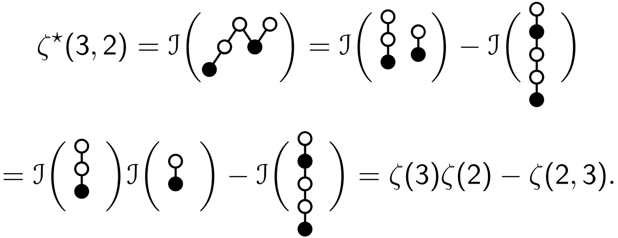
# Cut and subtract



A key move, which we call "cut and subtract", uses Property 2 in the form

$$\mathcal{I}(X_{a<b}) = \mathcal{I}(X) - \mathcal{I}(X_{b<a}),$$

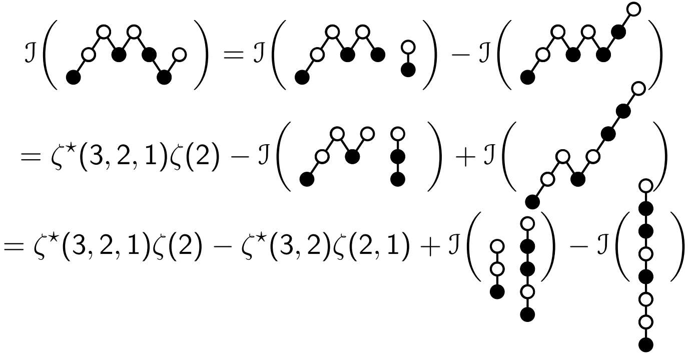
where $X$ is a disjoint union to which Property 1 can be applied. For example,



# The standard expansion



In fact, repeatedly using cut-and-subtract to "cut off" a chain on the right reduces any MZSV to a sum of products $\zeta^\star(u)\zeta(v)$ plus one or more MZVs. For example, $\zeta^\star(3,2,1,2)$ is



# The standard expansion cont'd



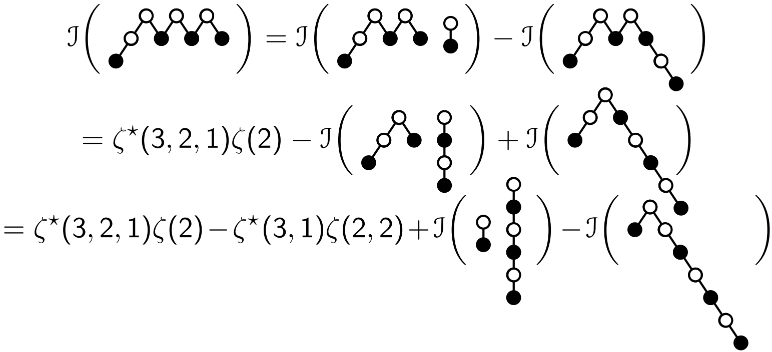
So $\zeta^\star(3,2,1,2)$ is

$$\zeta^\star(3,2,1)\zeta(2) - \zeta^\star(3,2)\zeta(2,1) + \zeta^\star(3)\zeta(2,1,2) - \zeta(2,1,2,3)$$

I call this the standard expansion. If the composition $I$ has last part greater than one, then the only term of the standard expansion of $\zeta^\star(I)$ that is an MZV is the last one, which in fact is $(-1)^{\ell(I)-1}\zeta(\bar{I})$. So $\zeta^\star(I)$ is a sum of decomposables plus $(-1)^{\ell(I)-1}\zeta(\bar{I})$, giving another proof of Eq. (8).

# The standard expansion cont'd

On the other hand, if the last part of $I$ is 1, cut-and-subtract proceeds a little differently. For example, the standard expansion of $\zeta^\star(3,2,2,1)$ is



# The standard expansion cont'd



$$= \zeta^\star(3,2,1)\zeta(2) - \zeta^\star(3,1)\zeta(2,2) + \zeta^\star(2)\zeta(2,2,2) - \mathcal{I}(P),$$

where $P$ is the "hanger" poset

$$\{\overline{x_0} < x_1 > x_2 > \overline{x_3} > x_4 > \overline{x_5} > x_6 > \overline{x_7}\}.$$

By application of Property 2 it can be seen that

$$\mathcal{I}(P) = \zeta(2,2,2,2) + 2\zeta(3,1,2,2) + 2\zeta(3,2,1,2) + 2\zeta(3,2,2,1).$$

In general an argument string ending in 1 can be written $wy$, $w \in \mathfrak{H}^0$, in the algebraic notation. Then the standard expansion of $\zeta^\star(wy)$ is a sum of terms $\zeta^\star(u)\zeta(v)$ plus or minus $\zeta(y \sqcup\!\sqcup\, w - yw)$.

# MZSVs of large depth



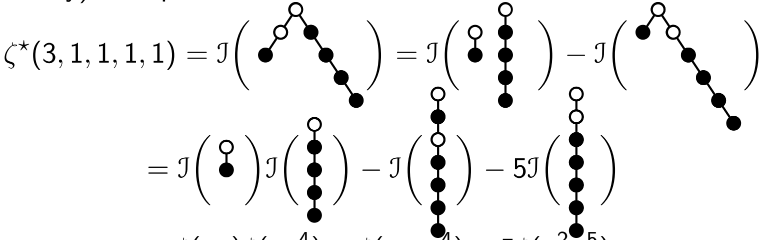
Recall that the collapsing sum of an MZSV of depth $d$ has $2^{d-1}$ terms. Thus, calculating an MZSV whose depth is close to its weight seems onerous. The highest depth is one less than the weight, and here the sum theorem comes to the rescue in evaluating the single MZSV, e.g.,

$$\zeta^{\star}(2,1,1,1,1,1) = \binom{6}{5}\zeta(7) = 6\zeta(7)$$

But what about an MZSV of next-to-greatest depth, e.g., $\zeta^{\star}(3,1,1,1,1)$? Its collapsing sum has 16 terms.

# MZSVs of large depth cont'd

But if we instead use the standard expansion, we can write it as a product minus a sum of MZVs of depth 5, hence (by duality) of depth 2:

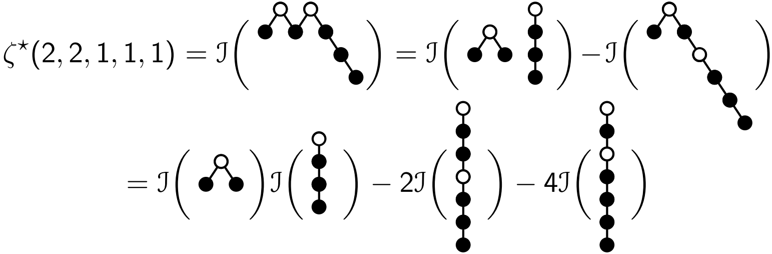


$$= \zeta(xy)\zeta(xy^4) - \zeta(xyxy^4) - 5\zeta(x^2y^5)$$
$$= \zeta(xy)\zeta(x^4y) - \zeta(x^4yxy) - 5\zeta(x^5y^2)$$
$$= \zeta(2)\zeta(5) - \zeta(5,2) - 5\zeta(6,1)$$
$$= \zeta(7) + \zeta(2,5) - 5\zeta(6,1),$$

which is far shorter than the 16-term collapsing sum!

# MZSVs of large depth cont'd

Here is another example:



$$=2\zeta(xy^2)\zeta(xy^3) - 2\zeta(xy^2xy^3) - 4\zeta(xyxy^4)$$

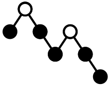
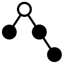
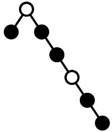
$$=2\zeta(x^2y)\zeta(x^3y) - 2\zeta(x^3yx^2y) - 4\zeta(x^4yxy)$$

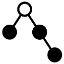
$$=2\zeta(3)\zeta(4) - 2\zeta(4,3) - 4\zeta(5,2)$$

$$=2\zeta(7) + 2\zeta(3,4) - 4\zeta(5,2).$$

# MZSVs of large depth cont'd

And yet another:

$\zeta^\star(2,1,2,1,1) = \mathcal{I}\Big($  $\Big) = \mathcal{I}\Big($   $\Big) - \mathcal{I}\Big($  $\Big)$

$= \mathcal{I}\Big($  $\Big)\mathcal{I}\Big($  $\Big) - 3\mathcal{I}\Big($  $\Big) - 3\mathcal{I}\Big($  $\Big)$

$$=3\zeta(xy^3)\zeta(xy^2) - 3\zeta(xy^3xy^2) - 3\zeta(xy^2xy^3)$$

$$=3\zeta(x^3y)\zeta(x^2y) - 3\zeta(x^2yx^3y) - 3\zeta(x^3yx^2y)$$

$$=3\zeta(4)\zeta(3) - 3\zeta(3,4) - 3\zeta(4,3)$$

$$=3\zeta(7).$$

# MZSVs of large depth cont'd



Calculations like these give simple formulas for all $n-2$ MZSVs of weight $n$ and depth $n-2$. The general result is

$$\zeta^\star(3,\{1\}_{n-3}) = \zeta(n) + \zeta(2,n-2) - (n-2)\zeta(n-1,1)$$

(where $\{1\}_p$ means $p$ repetitions of 1) and

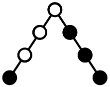
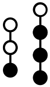
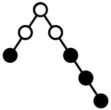
$$\zeta^\star(2,\{1\}_{r-1},2,\{1\}_{n-3-r}) = (r+1)\zeta(n) + (r+1)\zeta(r+2,n-2-r) - (n-2-r)\zeta(n-1-r,r+1) \quad (10)$$

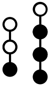
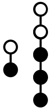
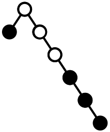
for $1 \le r \le n-3$. If $n \ge 5$ is odd there is exactly one case where the last two terms in Eq. (10) cancel, giving

$$\zeta^\star(2,\{1\}_{\frac{n-5}{2}},2,\{1\}_{\frac{n-3}{2}}) = \frac{n-1}{2}\zeta(n).$$

# MZSVs of large depth cont'd

The standard expansion continues to be useful at the next lower depth. For example,

$\zeta^\star(4,1,1,1) = \mathcal{I}\Big($  $\Big) = \mathcal{I}\Big($  $\Big) - \mathcal{I}\Big($  $\Big)$

$= \mathcal{I}\Big($  $\Big) - \mathcal{I}\Big($  $\Big) + \mathcal{I}\Big($  $\Big)$

$= \zeta(x^2y)\zeta(xy^3) - \zeta(xy)\zeta(x^2y^3) + \mathcal{I}\Big($  $\Big) + \mathcal{I}\Big($  $\Big) + 4\mathcal{I}\Big($  $\Big)$

# MZSVs of large depth cont'd



$$\begin{aligned}
&=\zeta(x^2y)\zeta(xy^3)-\zeta(xy)\zeta(x^2y^3)+\zeta(xyx^2y^3)+\zeta(x^2yxy^3)\\
&\quad+4\zeta(x^3y^4)\\
&=\zeta(x^2y)\zeta(x^3y)-\zeta(xy)\zeta(x^3y^2)+\zeta(x^3y^2xy)+\zeta(x^3yxy^2)\\
&\quad+4\zeta(x^4y^3)\\
&=\zeta(3)\zeta(4)-\zeta(2)\zeta(4,1)+\zeta(4,1,2)+\zeta(4,2,1)+4\zeta(5,1,1)\\
&=\zeta(7)+\zeta(3,4)-\zeta(6,1)-\zeta(2,4,1)+4\zeta(5,1,1).
\end{aligned}$$

This 5-term formula is better than the 8-term collapsing sum for $\zeta^\star(4,1,1,1)$. It is also better than the 10-term formula

$$\zeta^\star(4,1,1,1)=\sum_{a+b+c=6,\ a,b,c\geq 1} a\zeta(a+1,b,c) \qquad (11)$$

obtained by using Property 2 alone.

# Scaling by weight



The difference becomes more dramatic at higher weights (We've been limiting the weight so the diagrams fit on a slide). The standard expansion gives

$$\zeta^\star(4,\{1\}_{n-4}) = \zeta(n)+\zeta(3,n-3)-\zeta(n-1,1)-\zeta(2,n-3,1) + (n-3)\zeta(n-2,1,1),$$

in weight $n$, where $\{1\}_k$ means $k$ repetitions of 1. On the other hand, Eq. (11) generalizes to

$$\zeta^\star(4,\{1\}_{n-4}) = \sum_{a+b+c=n-1,\ a,b,c\geq 1} a\zeta(a+1,b,c)$$

with $\binom{n-2}{2}$ terms, and the collapsing sum for $\zeta^\star(4,\{1\}_{n-4})$ has $2^{n-4}$ terms. In weight 11, these numbers are 5, 36, and 128.

# Off to Bonn

I had just learned to do this type of calculation when my Spring 2020 sabbatical started. I had accepted an invitation to the Max-Planck-Institut für Mathematik (MPIM) in Bonn for the first five months of 2020. When I left at the end of 2019, I took along an MZV table for weight 11, copied from one produced by the Lille group. My idea was that I could use it to make test calculations of MZSVs using Yamamoto's representation. But my main project was a symmetry result on multiple $t$-values that I discussed in my talk in 2019. (This is now proved, and I hope to give a talk about that later).

# Settling in

I got to Bonn and settled in to my apartment, about 20 minutes' walk from downtown where the MPIM is located. Things functioned normally through January and February: lots of talks every week, tea each afternoon. I started to do some calculations with MZSVs, since I was planning to do joint work on the symmetry result with Steven Charlton, who would arrive in March. Indeed Steven arrived then, but I learned he'd taken a job in Hamburg starting in April. Something else happened in March: there were increasingly ominous reports about COVID-19 in Europe. I gave the last in-person Oberseminar talk at the MPIM on March 5. The next week pretty much all regular activities of the MPIM were suspended, as Germany went into lockdown.

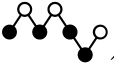
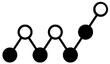
# Isolation

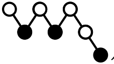


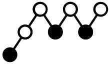
I was fortunate that I could still walk to my office in the MPIM every day, and use the library and computer facilities there. But otherwise everything was closed except for grocery stores, banks, post offices, and pharmacies. Meetings of more than two persons were prohibited. I could still meet with Steven once a week, and in fact he decided to stay in Bonn and do his lectures in Hamburg remotely. But otherwise my contacts were limited to e-mail, brief conversations with my landlady, and video calls to my wife and children. I was healthy and got enough exercise (since I avoided public transportation), but I had a lot of time on my hands!

# Pictorial proofs

I had already started writing a paper about the standard expansion and its implications. One thing I noticed was that some identities for MZSVs could be proved "pictorially" by a single cut-and-subtract move. For example,

$\zeta^\star(2,2,1,2) = \mathcal{I}\Big($  $\Big) = \mathcal{I}\Big($  $\Big) - \mathcal{I}\Big($  $\Big)$

$= \zeta^\star(2,2,1)\zeta^\star(2) - \mathcal{I}\Big($  $\Big)$

$= \zeta^\star(2,2,1)\zeta^\star(2) - \mathcal{I}\Big($  $\Big)$

$$= \zeta^\star(2,2,1)\zeta^\star(2) - \zeta^\star(3,2,2).$$

# Pictorial proofs cont'd



Here the vertical flip is just Property 3, and the horizontal flip follows since the two diagrams represent the same poset. This proof generalizes to show

$$\zeta^\star(\{2\}_m, 1, \{2\}_n) = \zeta^\star(\{2\}_m, 1)\zeta^\star(\{2\}_n) - \zeta^\star(\{2\}_{n-1}, 3, \{2\}_m)$$

for any positive integers $m, n$. Similar pictorial proofs show

$$\begin{aligned}\zeta^\star(\{2\}_m, 1, \{2\}_n, 1) &= \zeta^\star(\{2\}_m, 1)\zeta^\star(\{2\}_n, 1) \\ &\quad - \zeta^\star(\{2\}_n, 1, \{2\}_m, 1) \\ \zeta^\star(\{2\}_m, 3, \{2\}_n, 1) &= \zeta^\star(\{2\}_{m+1})\zeta^\star(\{2\}_{n+1}) \\ &\quad - \zeta^\star(\{2\}_n, 3, \{2\}_m, 1)\end{aligned}$$

These results appear in J. Zhao's book, but the proofs are not nearly so visual!

# Pictorial proofs cont'd



MZVs $\zeta(2,\dots,2,3,2,\dots,2)$ can be written in the form (Zagier, 2011)

$$c_0\zeta(2n+3)+\sum_{i=1}^{n} c_i\zeta(2i)\zeta(2n-2i+3),\ \ c_i\in\mathbb{Q}. \qquad (12)$$

Moreover, MZVs whose argument string is all 2's with a single 1 also have this form, since, e.g., $\zeta(2,2,3,2)=\zeta(2,2,1,2,2)$ by duality. The standard expansion shows that MZSVs $\zeta^\star(2,\dots,2,3,2,\dots,2)$ are also expressible in the form (12). Duality doesn't work for MZSVs, but the first pictorial identity, with Eqs. (1) and (2), implies

$$\zeta^\star(\{2\}_m,1,\{2\}_n)+\zeta^\star(\{2\}_{n-1},3,\{2\}_m)=(4-2^{3-2n})\zeta(2m+1)\zeta(2n),$$

so MZSVs $\zeta^\star(2,\dots,2,1,2,\dots,2)$ also have form (12).

# A new plan



So I continued working on my paper, as well as my joint work with Steven. But to stay sane, I embarked on a new project: to make a table of all $2^9 = 512$ weight-11 MZSVs in terms of the $P_{11} = 9$ quantities $\zeta(11)$, $\zeta(2)\zeta(9)$, $\zeta(3)\zeta(8)$, $\zeta(4)\zeta(7)$, $\zeta(5)\zeta(6)$, $\zeta(3)^2\zeta(5)$, $\zeta(2)\zeta(3)^3$, $\zeta(3)\zeta(6,2)$, and $\zeta(8,2,1)$. Using the pictorial technique described above, I would write out the standard expansion of each MZSV and then compute it using the weight-11 MZV table and MZSV tables for lower weights. Over the next three months or so I completed this project, producing a 65-page table. (You can decide yourself if this warded off insanity or was good evidence that I actually *did* go insane.)

# Complications



Naturally there were errors. I would get terms mixed up occasionally, and even found that I had incorrectly copied some of the relations from the tables of the Lille group. As a check I used the cyclic-derivation theorem of Ohno and Wakabayashi. If we define $\bar{C} : \mathfrak{H}^1 \to \mathfrak{H}^1 \otimes \mathfrak{H}^1$ to be the derivation taking $x$ to 0 and $y$ to $y \otimes x$, and $\bar{\mu} : \mathfrak{H}^1 \otimes \mathfrak{H}^1 \to \mathfrak{H}^1$ to be reversed concatenation, then the cylic derivation $C = \bar{\mu}\bar{C} : \mathfrak{H}^1 \to \mathfrak{H}^1$ has the property that $\zeta^\star(\tau C \tau(w)) = (n-1)\zeta(n)$ for any word of $\mathfrak{H}^1_{n-1}$ not a power of $y$. This has the effect of dividing up the depth-$d$ MZSVs of weight $n$ into equivalence classes, generally of size $n-d$, that sum to $(n-1)\zeta(n)$. Notice the equivalence classes get bigger as $d$ gets smaller (and bigger equivalence classes make it harder to track down errors).

# A longer stay



Already in January and February I had computed the depth-9 and depth-8 weight-11 MZSV's, but as the depth got smaller the going got harder. The reality of the pandemic set in. Plans for my family to visit during the Easter holiday had to be scrapped. In May there was limited reopening of stores, which was welcome since I needed new notebooks and pens for my calculations–and I had completely worn out a pair of shoes. My flight home at the end of May was cancelled, and there didn't even seem to be passenger flights out of the Bonn-Cologne airport for a time. The directors of the MPIM generously extended my stay for another three months, and I had no trouble extending the lease on my apartment (people who were supposed to come that summer had cancelled).

# Covid summer



It became possible to eat lunch in restaurants again, provided you filled out a form first. I got my *Aufenthaltserlaubnis* (residency permit) extended by a *Fiktionsbestätigung*, even though the agency that handles these things was officially closed. There were still no afternoon teas at the MPIM, but we would have afternoon meetings outside once a week. I continued my joint work with Steven. As spring turned to summer, my big calculation wound to a close. I finished my paper explaining and applying the standard expansion, and submitted it to the MPIM preprint series July 1. I finally got a ride home on August 6, on the emptiest trans-Atlantic flight I've ever taken. Then I had to wait two weeks before I was allowed into the Naval Academy.

# Back to Euler sums



To bring things back to Euler sums, consider sums of the form

$$\sum_{m=1}^{\infty} \frac{H_m^{e_1}(H_m^{(2)})^{e_2}\cdots(H_m^{(k)})^{e_k}}{m^p} \tag{13}$$

where $p \geq 2$, $H_m$ is the $m$th harmonic number, and

$$H_m^{(r)} = \sum_{j=1}^{m} \frac{1}{j^r}$$

is the $m$th generalized harmonic number. Then there is a simple formula for (13): if we think of $k^{e_k}\cdots 2^{e_2}1^{e_1}$ (where the exponents mean repetition) as a partition $\lambda$ of $n = e_1 + 2e_2 + \cdots + ke_k$, then (13) is

$$\sum_{\mu\leq\lambda}(-1)^{n-\ell(\lambda)}L_{\lambda,\mu}S_p(\mu).$$

# Back to Euler sums cont'd



Here $S_p(\mu)$ is the sum of all $\zeta^\star(p, I)$ such that the composition $I$ becomes the partition $\mu$ if we forget the order of the parts, e.g.,

$$S_p(32^2) = \zeta^\star(p, 2, 2, 3) + \zeta^\star(p, 2, 3, 2) + \zeta^\star(p, 3, 2, 2);$$

$\ell(\lambda)$ is the number of parts of $\lambda$; $\leq$ is the total order on partitions of a fixed integer given by reverse lexicographic order; and $(L_{\lambda,\mu})$ is the transition matrix from the basis $p_\lambda$ for Sym consisting of products of power-sum symmetric fuctions to the basis of monomial symmetric functions $m_\lambda$ (for which see Macdonald, *Symmetric Functions and Hall Polynomials*, pp. 102-103). In particular $L_{1^n,\lambda} = \binom{n}{\lambda}$ for any partition $\lambda$ of $n$, so we recover Eq. (7).

# Example



For example, since

$$p_1 p_3^2 = m_7 + m_{61} + 2m_{43} + 2m_{3^21}$$

we have

$$\sum_{n=1}^{\infty} \frac{H_n (H_n^{(3)})^2}{n^4} = S_4(7) - S_4(61) - 2S_4(43) + 2S_4(3^21)$$

From my table of weight-11 MZSVs,

$$S_4(7) = \frac{331}{2}\zeta(11) - 84\zeta(2)\zeta(9) - 20\zeta(4)\zeta(7) - 4\zeta(5)\zeta(6)$$

$$S_4(61) = \frac{2099}{16}\zeta(11) - 40\zeta(2)\zeta(9) - \frac{43}{6}\zeta(3)\zeta(8) - 33\zeta(4)\zeta(7) - \frac{35}{2}\zeta(5)\zeta(6) + \zeta(3)\zeta(6,2) + \zeta(8,2,1)$$

# Example cont'd



$$
\begin{aligned}
S_4(43) &= -\frac{10321}{24}\zeta(11) + 125\zeta(2)\zeta(9) + 55\zeta(3)\zeta(8) \\
&+ 108\zeta(4)\zeta(7) + \frac{325}{6}\zeta(5)\zeta(6) - 10\zeta(3)^2\zeta(5) \\
&- 10\zeta(8,2,1) \\
S_4(3^2 1) &= -\frac{65521}{48}\zeta(11) + \frac{15875}{36}\zeta(2)\zeta(9) + \frac{2077}{16}\zeta(3)\zeta(8) \\
&+ 320\zeta(4)\zeta(7) + \frac{1219}{8}\zeta(5)\zeta(6) - 15\zeta(3)^2\zeta(5) \\
&- \frac{2}{3}\zeta(2)\zeta(3)^3 - \frac{13}{4}\zeta(2)\zeta(6,2) - 27\zeta(8,2,1)
\end{aligned}
$$

# Example cont'd



Hence

$$\sum_{n=1}^{\infty} \frac{H_n (H_n^{(3)})^2}{n^4} =$$

$$-\frac{88111}{48}\zeta(11) + \frac{10583}{18}\zeta(2)\zeta(9) + \frac{3763}{24}\zeta(3)\zeta(8)$$

$$+ 437\zeta(4)\zeta(7) + \frac{2519}{12}\zeta(5)\zeta(6) - 10\zeta(3)^2\zeta(5) - \frac{4}{3}\zeta(2)\zeta(3)^3$$

$$- \frac{15}{2}\zeta(3)\zeta(6,2) - 35\zeta(8,2,1).$$