Formulas for Multiple Zeta-Star Values

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Outline

Introduction

This talk is about multiple zeta-star values, which are closely related to multiple zeta values. I'll give definitions. For positive integers $i_1, \dots, i_k$ with $i_1 > 1$, the corresponding multiple zeta value (MZV) is the real number

$$\zeta(i_1, i_2, \dots, i_k) = \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

The positive integer k is called the depth, and $i_1 + i_2 + \cdots + i_k$ is called the weight.

Introduction cont'd

Multiple zeta-star values (MZSVs) are defined similarly:

$$\zeta^\star(i_1, i_2, \dots, i_k) = \sum_{n_1 \ge n_2 \ge \cdots \ge n_k \ge 1} \frac{1}{n_1^{i_1} n_2^{i_2} \cdots n_k^{i_k}}.$$

The only difference from the previous definition is the non-strict inequalities in the summation indices. Clearly multiple zeta and multiple zeta-star values are related in a simple way: for example,

$$\zeta^\star(2,1,3) = \zeta(2,1,3) + \zeta(3,3) + \zeta(2,4) + \zeta(6).$$

and

$$\zeta(2,1,3) = \zeta^\star(2,1,3) - \zeta^\star(3,3) - \zeta^\star(2,4) + \zeta^\star(6).$$

Introduction cont'd

MZVs and MZSVs really go back to Euler for depth two, but interest in those of general depth took off in the 1990's with their simultaneous appearance in knot theory and theoretical physics. The first multiple zeta identity is $\zeta(2,1)=\zeta(3)$, i.e,

$$\sum_{n=2}^{\infty}\sum_{m=1}^{n-1}\frac{1}{n^2m}=\sum_{n=1}^{\infty}\frac{1}{n^3}.$$

This neat little identity is often rediscovered and posed as a problem. It has two remarkable generalizations: the sum theorem and the duality theorem. The sum theorem gives the sum of all multiple zeta values of weight n and a fixed depth as $\zeta(n)$. For example,

$$\zeta(4,1)+\zeta(3,2)+\zeta(2,3)=\zeta(3,1,1)+\zeta(2,2,1)+\zeta(2,1,2)=\zeta(5).$$

Introduction cont'd

The duality theorem gives a dual $\tau(I)$ for each sequence I so that $\zeta(I) = \zeta(\tau(I))$. The dual of (3) is $(2, 1)$, and the dual of $(2, 2, 1)$ is $(3, 2)$. There is a sum theorem for MZSVs, namely

$$\sum_{|I|=n,\ \ell(I)=k} \zeta^\star(I) = \binom{n-1}{k-1}\zeta(n).$$

But there is no nice analogue for the duality theorem. There are, however, many identities for MZSVs that have no analogues for MZVs. One such is

$$\zeta^\star(\{2\}_n, 1) = 2\zeta(2n+1), \tag{1}$$

where $\{2\}_n$ means n repetitions of 2.

Main Results

The preceding identity generalizes to MZSVs of the form $\zeta^\star(\{2\}_n, \{1\}_m)$. Here are our main results.

Theorem (1)

For any nonnegative integers $n > 1$ and $m \geq 0$, $\zeta^\star(\{2\}_n, \{1\}_m)$ is a rational polynomial in $\zeta(2), \zeta(3), \zeta(4), \ldots$.

This is *not* true for MZVs, e.g.,

$$\zeta(2,2,2,1,1) = \frac{1271}{72}\zeta(8) - 20\zeta(3)\zeta(5) + 3\zeta(2)\zeta(3)^2 + \frac{9}{2}\zeta(6,2).$$

Theorem (2)

For any positive integer n,

$$\zeta^\star(\{2\}_n, 1, 1) = (2n+1)\zeta(2n+2) - 2\sum^{n-1} \zeta(2i+1)\zeta(2n-2i+1).$$

Main Results cont'd

As we add more 1's the formulas get more complicated.

Theorem (3)

For any positive integer n,

$$\zeta^\star(\{2\}_n,1,1,1)=\frac{2}{3}\binom{2n+2}{2}\zeta(2n+3)-\sum_{i=2}^{n}\zeta(2i)\zeta(2n-2i+3)$$
$$+\sum_{\substack{i+j+k=n-3\\ i,j,k\ge 0}}\frac{4}{3}a(i,j,k)\zeta(2i+3)\zeta(2j+3)\zeta(2k+3),$$

where $a(i,j,k)$ is 1,3,6 if $\operatorname{card}\{i,j,k\}$ is respectively 1,2,3.

Methods

There are two main ingredients to the proofs of these results. The first is the ‘pictorial’ computations of MZSVs, based on Yamamoto’s representation of iterated integrals as labeled posets. The second is the symmetric sum theorem for MZVs, which I proved in 1992.

MZVs as iterated integrals

MZVs can be represented by iterated integrals as well as by series. For example,

$$\int_0^1 \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_1}{1-t_1} = \int_0^1 \frac{dt_3}{t_3} \int_0^{t_3} \sum_{i\ge 1} \frac{t_2^i}{i} \frac{dt_2}{1-t_2} = \int_0^1 \sum_{i,j\ge 1} \frac{t_3^{i+j}}{i(i+j)} \frac{dt_3}{t_3} = \sum_{i,j\ge 1} \frac{1}{i(i+j)^2} = \zeta(2,1).$$

We can represent such an integral by a monomial in noncommuting variables x and y: in this case, xy^2. But an even better representation is due to Shuji Yamamoto.

Yamamoto's formalism

Here is Yamamoto's idea. He considers iterated integrals over some subset of $[0,1]^n$, involving the two forms

$$\omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1-t}$$

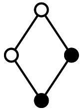
that we called x and y. Let (X,δ) be a 2-labeled poset, i.e., a finite partially ordered set X with a function $\delta : X \to \{0,1\}$. Call (X,δ) admissible if $\delta(x) = 1$ for all minimal $x \in X$ and $\delta(x) = 0$ for all maximal $x \in X$. For an admissible 2-labeled poset (X,δ), define the associated integral by

$$\mathcal{I}(X) = \int_{\Delta(X)} \prod_{x \in X} \omega_{\delta(x)}(t_x), \tag{2}$$

where $\Delta(X) = \{(t_x)_{x\in X} \in [0,1]^X \mid t_x < t_y \text{ if } x < y \text{ in } X\}$.

Graphical representation

We can represent a 2-labeled poset (X, δ) graphically by its Hasse diagram, with an open dot $\circ$ for those elements $x \in X$ with $\delta(x) = 0$, and a closed dot $\bullet$ for those $x \in X$ with $\delta(x) = 1$. For example, the 2-labeled poset $X = \{x_1, x_2, x_3, x_4\}$ with $x_1 > x_2 > x_4$, $x_1 > x_3 > x_4$, $\delta(x_1) = \delta(x_2) = 0$, and $\delta(x_3) = \delta(x_4) = 1$ has graphical representation



and associated integral

$$\mathcal{I}(X) = \int_{\substack{t_1 > t_2 > t_4 \\ t_1 > t_3 > t_4}} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \frac{dt_3}{1 - t_3} \frac{dt_4}{1 - t_4}.$$

2-labeled posets and MZVs

Admissibility of (X, δ) guarantees convergence of the integral $\mathcal{J}(X)$. If X is a chain, then $\mathcal{J}(X)$ is just the iterated integral representation of a multiple zeta value described above, e.g.

$\mathcal{J}\Big($  $\Big)$

$$= \int_{t_1<t_2<t_3<t_4<t_5<t_6} \frac{dt_6}{t_6}\frac{dt_5}{t_5}\frac{dt_4}{1-t_4}\frac{dt_3}{1-t_3}\frac{dt_2}{t_2}\frac{dt_1}{1-t_1} =$$

$$\int_0^1 \frac{dt_6}{t_6}\int_0^{t_6}\frac{dt_5}{t_5}\int_0^{t_5}\frac{dt_4}{1-t_4}\int_0^{t_4}\frac{dt_3}{1-t_3}\int_0^{t_3}\frac{dt_2}{t_2}\int_0^{t_2}\frac{dt_1}{1-t_1}$$
$$= \zeta(x^2y^2xy) = \zeta(3,1,2).$$

To get the algebraic description just read the chain from top to bottom, with x for open dots and y for closed ones.

Properties of the integral

Yamamoto proved the following theorem, which follows easily from general properties of iterated integrals.

Theorem (Yamamoto 2014)

Let X be an admissible 2-labeled poset.

1. *If Y is another admissible 2-labeled poset, $\mathcal{I}(X \amalg Y) = \mathcal{I}(X)\mathcal{I}(Y)$, where $\amalg$ is disjoint union.*
2. *If $a, b \in X$ are incomparable, let $X_{a<b}$ be X with the additional relation $a < b$. Then $\mathcal{I}(X) = \mathcal{I}(X_{a<b}) + \mathcal{I}(X_{b<a})$.*
3. *If $X^\vee$ is X with reversed order and new labeling function $\delta^\vee(x) = 1 - \delta(x)$, then $\mathcal{I}(X^\vee) = \mathcal{I}(X)$.*

Multiple zeta-star values

The multiple zeta-star values are easily written in Yamamoto's formalism; in fact

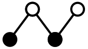
$$\zeta^\star(k_1, \dots, k_r) = \mathcal{J}(X),$$

where X is the poset

$$\{\bar{x}_1 < x_2 < \cdots < x_{k_1} > \bar{x}_{k_1+1} < x_{k_1+2} < \cdots < x_{k_1+k_2} > \cdots \\ > \bar{x}_{k_1+\cdots+k_{r-1}+1} < x_{k_1+\cdots+k_{r-1}+2} < \cdots < x_{k_1+\cdots+k_r}\},$$

with barred elements having label 1 and all others having label 0. For example,

$$\mathcal{J}\left(\right) = \int_{t_1<t_2>t_3<t_4} \frac{dt_4}{t_4}\frac{dt_3}{1-t_3}\frac{dt_2}{t_2}\frac{dt_1}{1-t_1}$$



Multiple zeta-star values cont'd

$$= \int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1-t_3} \int_{t_3}^1 \frac{dt_2}{t_2} \int_0^{t_2} \frac{dt_1}{1-t_1}$$

$$= \int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1-t_3} \int_{t_3}^1 \sum_{i\ge 1} \frac{t_2^{i-1}}{i} dt_2 =$$

$$\int_0^1 \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_3}{1-t_3} \sum_{i\ge 1} \frac{1-t_3^i}{i^2} = \int_0^1 \sum_{i,j\ge 1} \left[\frac{t_4^{j-1}}{i^2 j} - \frac{t_4^{i+j-1}}{i^2(i+j)} \right] dt_4$$

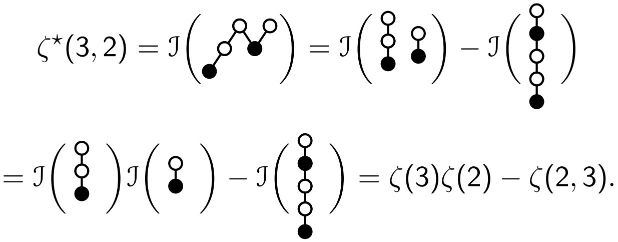
$$= \sum_{i=1}^\infty \frac{1}{i^2} \sum_{j=1}^\infty \left[\frac{1}{j^2} - \frac{1}{(i+j)^2} \right] = \sum_{i=1}^\infty \frac{1}{i^2} \sum_{j=1}^i \frac{1}{j^2} = \zeta^\star(2,2).$$

Cut and subtract

A key move, which we call "cut and subtract", uses Property 2 in the form

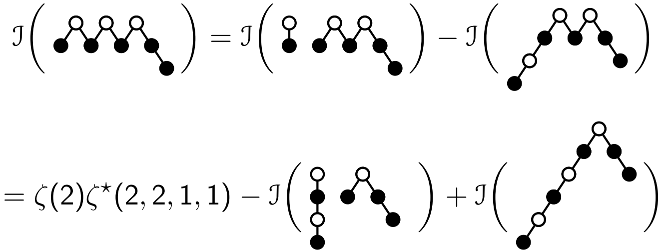
$$\mathcal{I}(X_{a<b}) = \mathcal{I}(X) - \mathcal{I}(X_{b<a}),$$

where X is a disjoint union to which Property 1 can be applied. For example,



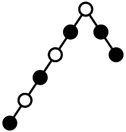
Reducing $\zeta^\star(\{2\}_n, \{1\}_m)$

In fact, by repeatedly using cut-and-subtract to "cut off" chains on the left reduces any MZSV of the form $\zeta^\star(\{2\}_n, \{1\}_m)$ to a sum of products of $\zeta(\{2\}_i)$ with simpler MZSVs of the same form, together with a term of a special type. For example, $\zeta^\star(2, 2, 2, 1, 1)$ is



Reducing $\zeta^\star(\{2\}_n, \{1\}_m)$ cont'd

$$= \zeta(2)\zeta^\star(2,2,1,1) - \zeta(2,2)\zeta^\star(2,1,1) + \mathcal{I}\left(\quad \right)$$



Now do a series of applications of Property 2 to reduce the last term to

$$3\zeta(4,2,2) + 3\zeta(2,4,2) + 3\zeta(2,2,4) + 4\zeta(3,3,2) + 4\zeta(3,2,3) + 4\zeta(2,3,3)$$

Analyzing these terms requires a study of symmetric sums of MZVs. But in fact I made such a study some years ago.

Symmetric Sum Theorem

The symmetric sum theorem reads as follows. Here S_n is the symmetric group on n letters, and $\mathfrak{P}([n])$ is the set of partitions of the set $[n] = \{1, 2, \ldots, n\}$.

Theorem (Hoffman 1992)

Let $m_1, \ldots, m_n$ be positive integers greater than 1. Then

$$\sum_{\sigma \in S_n} \zeta(m_{\sigma(1)}, \ldots, m_{\sigma(n)}) = \sum_{\{B_1,\ldots,B_k\} \in \mathfrak{P}([n])} (-1)^{n-k} \prod_{j=1}^{k} (\operatorname{card} B_j - 1)! \zeta(b_j),$$

where $b_i = \sum_{j \in B_i} m_j$.

First corollary

Two corollaries of the symmetric sum theorem are as follows. The first concerns the case where $m_1 = m_2 = \cdots = m_n$,

Corollary (1)

For positive integers $m \geq 2$,

$$n!\zeta(\{m\}_n) = \sum_{\{B_1,\dots,B_k\}\in\mathfrak{P}(n)} (-1)^{n-k} \prod_{j=1}^{k} (\operatorname{card} B_j - 1)!\zeta(m \operatorname{card} B_j)$$

It's possible to state this in terms of integer partitions, but the form above is most convenient for this talk.

Second corollary

A second corollary deals with the case where all but one of the n integers are the same. We will state it in the case where the smaller integer is 2.

Corollary (2)

For positive integers $a > 2$,

$$\sum_{i=1}^{n} \zeta(\{2\}_{i-1}, a, \{2\}_{n-i}) = \sum_{i=0}^{n-1} (-1)^i \zeta(a + 2i)\zeta(\{2\}_{n-1-i})$$

Notation

For integers $n_1, \ldots, n_k \geq 2$, we define $\mathrm{SymSum}\{n_1, \ldots, n_k\}$ to be the symmetrization of $\zeta(n_1, \ldots, n_k)$, i.e.,

$$\frac{1}{\mathrm{card}\, G} \sum_{\sigma \in S_k} \zeta(n_{\sigma(1)}, \ldots, n_{\sigma(k)}),$$

where G is the isotropy group of $(n_1, \ldots, n_k)$, i.e., the permutations σ such that $n_{\sigma(i)} = n_i$ for all i. Thus, e.g., $\mathrm{SymSum}\{\{2\}_k\} = \zeta(\{2\}_k)$ and

$$\mathrm{SymSum}\{4, \{2\}_{n-1}\} = \sum_{j=0}^{n} \zeta(\{2\}_j, 4, \{2\}_{n-1-j}).$$

More symmetric sums

From Corollary 2 it follows that

$$\mathrm{SymSum}\{4, \{2\}_{n-1}\} = \sum_{j=0}^{n-1} (-1)^j \zeta(4+2j)\zeta(\{2\}_{n-1-j})$$

Similarly

$$\mathrm{SymSum}\{6, \{2\}_{n-1}\} = \sum_{j=0}^{n-1} (-1)^j \zeta(6+2j)\zeta(\{2\}_{n-1-j})$$

and so forth.

More symmetric sums cont'd

If $\mathrm{SymSum}\{3,3,\{2\}_{n-2}\}$ is written as a polynomial in the $\zeta(i)$, all terms that don't have a factor $\zeta(i)$, i odd, will be rational multiples of $\zeta(2n+2)$. We will write the sum of such terms as $\mathrm{SymSum}^0\{3,3,\{2\}_{n-2}\}$. Then

$$\begin{aligned}\zeta(3)\,\mathrm{SymSum}\{3,\{2\}_{n-2}\} &= 2\,\mathrm{SymSum}\{3,3,\{2\}_{n-2}\} \\ &+ \mathrm{SymSum}\{6,\{2\}_{n-2}\} + \mathrm{SymSum}\{5,3,\{2\}_{n-3}\}\end{aligned}$$

$$\begin{aligned}\zeta(5)\,\mathrm{SymSum}\{3,\{2\}_{n-3}\} &= \mathrm{SymSum}\{5,3,\{2\}_{n-3}\} \\ &+ \mathrm{SymSum}\{8,\{2\}_{n-3}\} + \mathrm{SymSum}\{7,3,\{2\}_{n-4}\}\end{aligned}$$

and so on, from which follows

More symmetric sums cont'd

$$2\,\mathrm{SymSum}^0\{3,3,\{2\}_{n-2}\} = \sum_{i=0}^{n-2}(-1)^{i+1}\,\mathrm{SymSum}\{6{+}2i,\{2\}_{n-2-i}\}$$

Applying Corollary 2, we have

$$\begin{aligned}
&2\,\mathrm{SymSum}^0\{3,3,\{2\}_{n-2}\} = \\
&\quad \sum_{i=0}^{n-2}\sum_{j=0}^{n-i-2}(-1)^{i+j+1}\zeta(6+2i+2j,\{2\}_{n-2-i-j}) \\
&\quad = \sum_{k=0}^{n-2}(-1)^{k+1}(k+1)\zeta(6+2k,\{2\}_{n-2-k}) \\
&\quad = \sum_{k=0}^{n-1}(-1)^{k}k\zeta(4+2k,\{2\}_{n-1-k}).
\end{aligned}$$

Third corollary

This gives us our third corollary.

Corollary (3)

$$\mathrm{SymSum}^0\{3,3,\{2\}_{m-2}\} = \frac{1}{2}\sum_{i=0}^{m-1}(-1)^i i\zeta(4+2i)\zeta(\{2\}_{m-1-i})$$

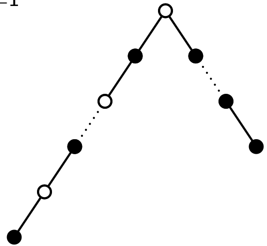
This result, which I only proved in recent weeks, is key to our proofs of Theorems 2 and 3.

Back to $\zeta^\star(\{2\}_n, \{1\}_m)$

By use of cut-and-subtract moves, we have

$$\zeta^\star(\{2\}_n, \{1\}_m) = \sum_{i=1}^{n-1} (-1)^{i-1} \zeta(\{2\}_i) \zeta^\star(\{2\}_{n-i}, \{1\}_m)$$

$$+(-1)^{n-1} \mathcal{I}\Bigg(\quad \Bigg),$$



where there are n black dots on the ascending diagonal and m black dots on the descending diagonal.

Back to $\zeta^\star(\{2\}_n, \{1\}_m)$ cont'd

Using Property 2 of Yamamoto's representation, we can reduce the last term to

$$(-1)^{n-1} \sum_{\substack{a_1+a_2+\cdots+a_n=n+m \\ a_i \geq 0}} a_1 a_2 \cdots a_n \zeta(a_1+1, a_2+1, \ldots, a_n+1),$$

which can be seen to be a symmetric sum and so, by the symmetric sum theorem, a polynomial in $\zeta(i)$, $i \geq 2$. This allows us to prove Theorem 1 by induction on n.

The basic equation

By writing the expression on the preceding slide explicitly as symmetric sums, we have our basic equation for $\zeta^\star(\{2\}_n, \{1\}_m)$.

$$\zeta^\star(\{2\}_n, \{1\}_m) = \sum_{i=1}^{n-1} (-1)^{i-1} \zeta(\{2\}_i) \zeta^\star(\{2\}_{n-i}, \{1\}_m) +$$

$$(-1)^{n-1} \sum_{\substack{2 \le i_1 \le \cdots \le i_n \\ i_1 + \cdots + i_n = 2n+m}} (i_1 - 1) \cdots (i_n - 1) \operatorname{SymSum}\{i_1, \dots, i_n\}$$

The case $m = 2$

In the case $m = 2$ the basic equation becomes

$$\zeta^\star(\{2\}_n, 1, 1) = \sum_{i=1}^{n-1} (-1)^{i-1} \zeta(\{2\}_i) \zeta^\star(\{2\}_{n-i}, 1, 1) +$$
$$(-1)^{n-1}(3\,\mathrm{SymSum}\{4, \{2\}_{n-1}\} + 4\,\mathrm{SymSum}\{3, 3, \{2\}_{n-2}\})$$

To prove Theorem 2, we use induction on n, the case $\zeta^\star(2, 1, 1) = 3\zeta(4)$ being well-known.

The case $m = 2$ cont'd

Applying the induction hypothesis gives

$$\zeta^\star(\{2\}_n,1,1) = \sum_{i=1}^{n-1}(-1)^{i-1}(2n-2i+1)\zeta(\{2\}_i)\zeta(2n-2i+2)$$

$$+2\sum_{i=1}^{n-1}\sum_{j=1}^{n-1-i}(-1)^i\zeta(\{2\}_i)\zeta(2j+1)\zeta(2n-2i-2j+1)$$

$$+3(-1)^{n-1}\,\mathrm{SymSum}\{4,\{2\}_{n-1})+4(-1)^{n-1}\,\mathrm{SymSum}\{3,3,\{2\}_{n-2}\}$$

Note that terms of the form $\zeta(2i+1)\zeta(2n-2i+1)$ can only appear as part of the second symmetric sum, where they arise from partitions of $\{3,3,2,\dots,2\}$ into two parts, each of which has a 3.

The case $m = 2$ cont'd

That is, from a partition of the form

$$\{\{3, \{2\}_{i-1}\}, \{3, \{2\}_{n-i-1}\}\}.$$

By the symmetric sum theorem, the coefficient of $\zeta(2i+1)\zeta(2n-2i+1)$ in $\text{SymSum}\{3, 3, \{2\}_{n-2}\}$ is

$$\frac{1}{2(n-2)!}(-1)^{n-2}\binom{n-2}{i-1}(i-1)!(n-i-1)! = (-1)^n \frac{1}{2},$$

which agrees with the conclusion.

The case $m = 2$ cont'd

Next consider terms of the form

$$\zeta(\{2\}_i)\zeta(2j+1)\zeta(2n-2i-2j+1) \tag{3}$$

with $i > 0$. Such a term occurs in the double sum with coefficient $2(-1)^i$. The term also arises in the second symmetric sum from partitions of the form

$$\{\{3,\{2\}_{j-1}\},\{3,\{2\}_{n-i-j-1}\},B_1,\dots,B_k\}$$

where $B_1 \cup \dots \cup B_k = \{\{2\}_i\}$. The coefficient of (3) in $\mathrm{SymSum}\{3,3,\{2\}_{n-2}\}$ is

The case $m = 2$ cont'd

$$\sum_{\substack{\text{partitions}\\ B_1\cup\cdots\cup B_k\\ \text{of}\,\{\{2\}_i\}}} \frac{(-1)^{k+2-n}}{2(n-2)!}\binom{n-2}{i}\binom{n-2-i}{j-1}(j-1)!(n-i-j-1)!$$

$$\times \frac{1}{\zeta(\{2\}_i)}\prod_{l=1}^{k}(\operatorname{card} B_l - 1)!\zeta(2\operatorname{card} B_l)$$

which by Corollary 1 is $(-1)^{i+n}\frac{1}{2}$, so that the term from $4(-1)^{n-1}\operatorname{SymSum}\{3,3,\{2\}_{n-2}\}$ cancels the one from the double sum.

The case $m = 2$ cont'd

Finally, we consider the the coefficient of $\zeta(2n+2)$. We must show that

$$(2n+1)\zeta(2n+2) = \sum_{i=1}^{n-1}(-1)^{i-1}\zeta(\{2\}_i)(2n-2i+1)\zeta(2n-2i+2)+$$
$$3(-1)^{n-1}\operatorname{SymSum}\{4,\{2\}_{n-1}\}+4(-1)^{n-1}\operatorname{SymSum}^0\{3,3,\{2\}_{n-2}\}$$

After rearrangement and division by $(-1)^{n-1}$, this is

$$\sum_{i=0}^{n-1}(-1)^{i}(2n+3)\zeta(2i+4)\zeta(\{2\}_{n-1-i}) = 3\operatorname{SymSum}\{4,\{2\}_{n-1}\} + 4\operatorname{SymSum}^0\{3,3,\{2\}_{n-2}\},$$

which follows from Corollaries 2 and 3.

The case $m = 3$

In the case $m = 3$ the basic equation becomes

$$\zeta^\star(\{2\}_n, 1, 1, 1) = \sum_{i=1}^{n-1} \zeta(\{2\}_i)\zeta^\star(\{2\}_{n-i}, 1, 1, 1) + (-1)^{n-1}(4\,\mathrm{SymSum}\{5, \{2\}_{n-1}\} + 6\,\mathrm{SymSum}\{4, 3, \{2\}_{n-2}\} + 8\,\mathrm{SymSum}\{3, 3, 3, \{2\}_{n-3}\}) \quad (4)$$

To prove Theorem 3, we need to show that the coefficient of $\zeta(2n+3)$ is $\frac{2}{3}\binom{2n+2}{2}$; that

$$\text{coeff. of } \zeta(2i)\zeta(2n+3-2i) = \begin{cases} 0, & i = 1, \\ 2 - 4i, & i \geq 2; \end{cases}$$

The case $m = 3$ cont'd

that, for nonnegative integers i, j, k whose sum is $n-3$, the coefficient of $\zeta(3+2i)\zeta(3+2j)\zeta(3+2k)$ is $\frac{4}{3}a(i,j,k)$; and that, if i, j, k are nonnegative integers whose sum is less than $n-3$, then the coefficient of

$$\zeta(3+2i)\zeta(3+2j)\zeta(3+2k)\zeta(2(n-3-i-j-k))$$

is zero. (Recall that $a(i,j,k)$ is 1,3, or 6 if $\operatorname{card}\{i,j,k\}$ is 1,2, or 3 respectively.)

Coefficient of $\zeta(2n+3)$

Note that $\zeta(2n+3)$ cannot occur in the first $n-1$ terms of (4). The coefficient of $\zeta(2n+3)$ in $\mathrm{SymSum}\{5,\{2\}_{n-1}\}$ is $\frac{1}{(n-1)!}(-1)^{n-1}(n-1)! = (-1)^{n-1}$. The coefficient of $\zeta(2n+3)$ in $\mathrm{SymSum}\{4,3,\{2\}_{n-2}\}$ is $\frac{1}{(n-2)!}(-1)^{n-1}(n-1)! = (-1)^{n-1}(n-1)$, and the coefficient of $\zeta(2n+3)$ in $\mathrm{SymSum}\{3,3,3,\{2\}_{n-3}\}$ is

$$\frac{1}{3!(n-3)!}(-1)^{n-1}(n-1)! = (-1)^{n-1}\frac{(n-1)(n-2)}{6}.$$

So the coefficient of $\zeta(2n+3)$ is

$$4 + 6(n-1) + 8\frac{(n-1)(n-2)}{6} = \frac{2}{3}\binom{2n+2}{2}.$$

Coefficient of $\zeta(2)\zeta(2n+1)$

There is a contribution of $\frac{2}{3}\binom{2n}{2}$ to the coefficient of $\zeta(2)\zeta(2n+1)$ from the first term of (4). The coefficient of $\zeta(2)\zeta(2n+1)$ in $\mathrm{SymSum}\{5,\{2\}_{n-1}\}$ is $\frac{(-1)^n}{(n-1)!}\binom{n-1}{1}(n-2)! = (-1)^n$, that in $\mathrm{SymSum}\{4,3,\{2\}_{n-2}\}$ is $\frac{(-1)^n}{(n-2)!}\binom{n-2}{1}(n-2)! = (-1)^n(n-2)$, and that in $\mathrm{SymSum}\{3,3,3,\{2\}_{n-3}\}$ is $\frac{(-1)^n}{3!(n-3)!}\binom{n-3}{1}(n-2)! = (-1)^n\frac{(n-2)(n-3)}{6}$. So in all the coefficient of $\zeta(2)\zeta(2n+1)$ is

$$\frac{2}{3}\binom{2n}{2} - 4 - 6(n-1) - \frac{4}{3}(n-2)(n-3) = 0$$

Coefficient of $\zeta(4)\zeta(2n-1)$

Since $\zeta(2,2) = \frac{3}{4}\zeta(4)$, there is a contribution of

$$-\frac{3}{4}\cdot\frac{2}{3}\binom{2n-2}{2} = -\frac{2n-5n+3}{2}$$

from the second term of (4). Contributions to this coefficient in $\mathrm{SymSum}\{5,\{2\}_{n-1}\}$ come from partitions of the form $\{\{5,\{2\}_{n-3}\},\{2\},\{2\}\}$ and $\{\{5,\{2\}_{n-3}\},\{2,2\}\}$, which by Corollary 1 together contribute $(-1)^{n-1}\frac{3}{4}$. Similarly, in $\mathrm{SymSum}\{4,3,\{2\}_{n-2}\}$ partitions of form $\{\{4,3,\{2\}_{n-4}\},\{2\},\{2\}\}$ and $\{\{4,3,\{2\}_{n-4}\},\{2,2\}\}$ together contribute $(-1)^{n-1}\frac{3}{4}(n-3)$ to the coefficient of $\zeta(4)\zeta(2n-1)$. But there is also a contribution of $(-1)^n$ from the partition $\{\{4\},\{3,\{2\}_{n-2}\}\}$.

Coefficient of $\zeta(4)\zeta(2n-1)$ cont'd

In SymSum$\{3,3,3,\{2\}_{n-3}\}$ there are contributions from partitions of the form $\{\{\{3\}_3,\{2\}_{n-5}\},\{2\},\{2\}\}$ and $\{\{\{3\}_3,\{2\}_{n-5}\},\{2,2\}\}$, which in all give $(-1)^{n-1}\frac{(n-3)(n-4)}{8}$. Summing the contributions gives

$$-\frac{2n^2-5n+3}{2}+3+\frac{9(n-3)}{2}-6+(n-3)(n-4)=-6.$$

Coefficient of $\zeta(2k)\zeta(2n-2k+3)$

We can now use induction on k to show that the coefficient of $\zeta(2k)\zeta(2n-2k+3)$ in (4) is $2-4k$ for $k \geq 2$, using the preceding coefficient as the base case. By the induction hypothesis, the first $2k$ terms of (4) contribute

$$\zeta(2)(6-4k)\zeta(2k-2) - \zeta(2,2)(10-4k)\zeta(2k-4) + \cdots + (-1)^{k-3}\zeta(\{2\}_{k-2})(-6)\zeta(4) + (-1)^{k-1}\frac{2}{3}\binom{2n-2k+1}{2}\zeta(2n-2k+3). \quad (5)$$

Now we look at contributions from the symmetric sums.

Coefficient of $\zeta(2k)\zeta(2n-2k+3)$ cont'd

The contribution from $4(-1)^{n-1}\,\mathrm{SymSum}\{5,\{2\}_{n-1}\}$ can be seen from Corollary 1 to be

$$4(-1)^k\zeta(\{2\}_k)\zeta(2n-2k+3) \tag{6}$$

The contribution from $6(-1)^{n-1}\,\mathrm{SymSum}\{4,3,\{2\}_{n-2}\}$ comes in two parts–that from partitions in which 4 and 3 are in the same block, which add up to

$$6(-1)^k(n-k-1)\zeta(\{2\}_k)\zeta(2n-2k+3), \tag{7}$$

and those in which they are in different blocks, giving

$$6(-1)^k\,\mathrm{SymSum}\{4,\{2\}_{k-2}\}\zeta(2n-2k+3). \tag{8}$$

Coefficient of $\zeta(2k)\zeta(2n-2k+3)$ cont'd

The contribution from $8(-1)^{k-1}\,\mathrm{SymSum}\{3,3,3,\{2\}_{n-2}\}$ comes from partitions in which all the 3's are in the same block, giving

$$8(-1)^k \frac{(n-k-1)(n-k-2)}{6}\zeta(\{2\}_k)\zeta(2n-2k+3) \qquad (9)$$

and from partitions in which only two 3's are in a block, giving

$$8(-1)^k\,\mathrm{SymSum}^0\{3,3,\{2\}_{k-3}\}\zeta(2n-2k+3) \qquad (10)$$

Now note that the sum of (6), (7) and (9) cancels the last term of (5).

Coefficient of $\zeta(2k)\zeta(2n-2k+3)$ cont'd

Next, use Corollaries 2 and 3 to show that the sum of (8) and (10) is

$$(-1)^{k-1}\zeta(2n-2k+3)\sum_{i=0}^{k-2}(-1)^i(6+4i)\zeta(4+2i)\zeta(\{2\}_{k-2-i})$$

The first $k-2$ terms cancel the remaining terms of (5), leaving only the last term

$$\begin{aligned}(-1)^{k-1}\zeta(2n-2k+3)(-1)^{k-2}(6+4k-8)\zeta(4+2k-4)\\ =(2-4k)\zeta(2k)\zeta(2n-2k+3).\end{aligned}$$

Completing the proof

Now note that for i, j, k summing to $n-3$, terms of form $\zeta(3+2i)\zeta(3+2j)\zeta(3+2k)$ can only arise in (4) from the third symmetric sum. They come from partitions of form $\{\{3,\{2\}_i\},\{3,\{2\}_j\},\{3,\{2\}_k\}\}$ and add up to

$$(-1)^{n-1}\frac{1}{3!(n-3)!}\binom{n-3}{i\ j\ k}i!j!k!a(i,j,k) = (-1)^{n-1}\frac{a(i,j,k)}{6},$$

which times $8(-1)^{n-1}$ is $\frac{4}{3}a(i,j,k)$.

Completing the proof cont'd

Finally, we have to show that if $i + j + k < n - 3$, the contributions to the coefficient of

$$\zeta(3+2i)\zeta(3+2j)\zeta(3+2k)\zeta(2(n-i-j-k)$$

from the first $n-1$ terms of (4) cancel out contributions from $(-1)^{n-1} 8\,\mathrm{SymSum}\{3,3,3,\{2\}_{n-3}\}$. This can be done using Corollary 1.