\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \]

\[ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = f v - \frac{1}{\rho_0} \frac{\partial p}{\partial x} + \]

\[ \frac{\partial}{\partial x}(A_H \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(A_H \frac{\partial u}{\partial y}) + \frac{\partial}{\partial z}(A_V \frac{\partial u}{\partial z}) \]

\[ \frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} = -f u - \frac{1}{\rho_0} \frac{\partial p}{\partial y} + \]

\[ \frac{\partial}{\partial x}(A_H \frac{\partial v}{\partial x}) + \frac{\partial}{\partial y}(A_H \frac{\partial v}{\partial y}) + \frac{\partial}{\partial z}(A_V \frac{\partial v}{\partial z}) \]

\[ \frac{\partial p}{\partial z} = -\rho g \]

\[ \frac{\partial T}{\partial t} + \frac{\partial u T}{\partial x} + \frac{\partial v T}{\partial y} + \frac{\partial w T}{\partial z} = \frac{\partial}{\partial x}(K_H \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(K_H \frac{\partial T}{\partial y}) + \]

\[ \frac{\partial}{\partial z}(K_V \frac{\partial T}{\partial z}) \]
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]
\[
\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} = f v - \frac{1}{\rho_0} \frac{\partial p}{\partial x} + \frac{\partial}{\partial y}(A_H \frac{\partial u}{\partial x}) + \frac{\partial}{\partial z}(A_V \frac{\partial u}{\partial z})
\]
\[
\frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} = -f u - \frac{1}{\rho_0} \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(A_H \frac{\partial v}{\partial x}) + \frac{\partial}{\partial z}(A_V \frac{\partial v}{\partial z})
\]
\[
\frac{\partial p}{\partial z} = -\rho g
\]
\[
\frac{\partial T}{\partial t} + \frac{\partial uT}{\partial x} + \frac{\partial vT}{\partial y} + \frac{\partial wT}{\partial z} = \frac{\partial}{\partial x}(K_H \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(K_H \frac{\partial T}{\partial y}) + \frac{\partial}{\partial z}(K_V \frac{\partial T}{\partial z})
\]
\[
\frac{\partial S}{\partial t} + \frac{\partial uS}{\partial x} + \frac{\partial vS}{\partial y} + \frac{\partial wS}{\partial z} = \frac{\partial}{\partial x}(K_H \frac{\partial S}{\partial x}) + \frac{\partial}{\partial y}(K_H \frac{\partial S}{\partial y}) + \frac{\partial}{\partial z}(K_V \frac{\partial S}{\partial z})
\]
\[
\rho = \rho(T, S) = \frac{P}{\alpha + 0.698P}
\]
where
\[
P = 5890 + 38T - 0.375T^2 + 3S,
\]
\[
\alpha = 1779.5 + 11.25T - 0.0745T^2 - (3.8 + 0.01T)S
\]
Steady 2D Flows: Stream function $\psi(x, y)$;

$$\frac{dx}{dt} = \frac{\partial \psi}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial \psi}{\partial x}$$

Fact: **Instantaneous stream functions (contours of $\psi$) are particle trajectories.**

Strategy:

Determine stability of stagnation (equilibrium) points

Find directions of compression and stretching (stable and unstable manifolds)

Strategy works for very special time-dependent systems (periodic perturbations), but fails for general time-dependent vector fields.
\[ \psi(x, y, t) = \frac{A}{k} \sin \pi y \sin kx + k \epsilon \cos \omega t \cos kx \]
Duffing’s Equation

\[ \frac{dx}{dt} = y, \quad \frac{dy}{dt} = x - x^3 + \epsilon \sin t \]

The flow is incompressible (\( \text{div} \, \mathbf{v} = 0 \)) so there is a streamfunction \( \psi(x, y, t) \) such that

\[ v_1 = -\frac{\partial \psi}{\partial y}, \quad v_2 = \frac{\partial \psi}{\partial x} \]

\[ \psi = -\frac{1}{2} y^2 + \frac{1}{2} x^2 - \frac{1}{4} x^4 + \epsilon x \sin t \]