A Note on Parametrization

The key to parametrization is to realize that the goal of this method is to describe the location of all points on a geometric object, a curve, a surface, or a region. This description must be one-to-one and onto: every point must be described once and only once.

1 Parametrization of Curves in $R^2$

Let us begin with parametrizing the curve $C$ whose equation is given by

$$x^2 + y^2 = 4$$

i.e., a circle of radius 2 centered at the origin. We start by associating a position vector $r$ to each point $(x, y)$ on $C$ through the relation

$$r = \langle x, y \rangle.$$ 

The coordinates $x$ and $y$ in (2) are not arbitrary -- they are related through equation (1). This means that we are free to assign a value to only one of the coordinates of a typical point on $C$; the other coordinate must be determined from the equation of the circle. For this reason we say $C$ has one degree of freedom.

Choosing $x$ as the parameter for $C$, we see from (1) that

$$y = \pm \sqrt{4 - x^2},$$

where the positive square root describes those points on $C$ that lie above the $x$-axis and the negative square root the points below the $x$-axis. The complete parametrization of $C$ is

$$r_1(x) = \langle x, \sqrt{4 - x^2} \rangle \quad \text{and} \quad r_2(x) = \langle x, -\sqrt{4 - x^2} \rangle,$$

where $-2 \leq x \leq 2$ for $r_1$ and $-2 < x < 2$ for $r_2$. Note that the points $(-2,0)$ and $(2,0)$ are arbitrarily assigned to $r_1$. We can now use the parametrization of $C$ to determine tangent vectors to $C$, plot $C$ on a graphics software, or to perform a line integral around $C$.

Although the parametrization in (3) is adequate for the purpose of describing $C$, it is not the most convenient description of this curve. A more efficient way to view $C$ is to use polar coordinates to describe its points: $x = 2\cos\theta, y = 2\sin\theta$, with $\theta \in [0, 2\pi)$. So $C$ can also be parametrized as

$$r_3(\theta) = \langle 2\cos\theta, 2\sin\theta \rangle, \quad \theta \in [0, 2\pi).$$

Note that $r_3$ in (4) does the job of both $r_1$ and $r_2$ in (3).

The parametrizations $r_1$, $r_2$ and $r_3$ are just a few ways out of the infinitely many ways that one could describe $C$. Here are three other parametrizations of the same curve:

$$r_4(t) = \langle 2\sin t, 2\cos t \rangle, \quad t \in [0, 2\pi),$$

(5)
where $C$ is traversed in the clockwise direction,

$$r_5(u) = (-2 \sin u, 2 \cos u), \quad u \in [0, 2\pi),$$

where $C$ is traversed in the counterclockwise direction (how is $r_5$ different from $r_3$?) and

$$r_6(w) = (2 \sin 2w, 2 \cos 2w), \quad w \in [0, \pi).$$

To understand the difference between $r_4$ and $r_6$, compute the speed of a particle traveling around $C$ according to these parametrizations.

Let us now consider parametrizations of other familiar curves. Any two dimensional curve whose equation is given by $y = f(x)$ can be parametrized as

$$r(x) = \langle x, f(x) \rangle, \quad x \in (a, b),$$

so, for instance, the straight line $y = mx + b$ can be viewed as

$$r(x) = \langle x, mx + b \rangle.$$

The circle of radius $a$ centered at $(b, c)$ is parametrized as

$$r(\theta) = \langle b + a \cos \theta, c + a \sin \theta \rangle, \quad \theta \in (0, 2\pi].$$

The ellipse whose equation is given by $a^2 x^2 + b^2 y^2 = c^2$ is parametrized as (to see where the following expressions come from, divide $a^2 x^2 + b^2 y^2 = c^2$ by $c^2$ and set the term containing $x^2$ equal to $\cos^2 t$ and the one containing $y^2$ to $\sin^2 t$)

$$r(t) = \langle \frac{c}{a} \cos t, \frac{c}{b} \sin t \rangle \quad t \in (0, 2\pi).$$

2 Parametrization of Curves in $R^3$

Similar to curves in $R^2$, curves in $R^3$ still have only one degree of freedom, that is, a single parameter is sufficient to describe the coordinates of a typical point on curves in $R^3$. As an example, consider the straight line $C$ that connects the two points $P = (1, 2, 1)$ and $Q = (-1, 1, 3)$. Let $P = (1, 2, 1)$ and $Q = (-1, 1, 3)$. Define $v = Q - P = (-2, -1, 2)$. Note that $v$ is parallel to the line $C$. So every point $S$ on $C$ can be accessed by the vector

$$S = P + tv$$

for some $t \in R$. So

$$r(t) = \langle 1, 2, 1 \rangle + t\langle -2, -1, 2 \rangle, \quad t \in R$$

is a parametrization of $C$. In terms of coordinates, (12) is equivalent to

$$\begin{align*}
x(t) &= 1 - 2t \\
y(t) &= 2 - t \\
z(t) &= 1 + 2t
\end{align*}$$
Every straight line \( C \), whether in \( R^2 \) or \( R^3 \), can be parametrized as

\[
\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \quad t \in \mathbb{R}
\]

where \( \mathbf{r}_0 \) is the position vector corresponding to a known point on \( C \) (such as \( \langle 1, 2, 1 \rangle \) in our previous example), and \( \mathbf{v} \) is a vector parallel to \( C \). For instance, to find the parametrization of the line of intersection between the two planes \( 2x - 3y + z = 2 \) and \( x + y + z = 0 \), first we find a point on this line by setting \( z = 0 \) in the equations of the planes and then solve for \( x \) and \( y \) to see that \( \left( \frac{2}{5}, -\frac{2}{5}, 0 \right) \) lies on \( C \). Next, we note that the vectors \( \mathbf{n}_1 = \langle 2, -3, 1 \rangle \) and \( \mathbf{n}_2 = \langle 1, 1, 1 \rangle \) are normal to the planes. Therefore,

\[
\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -4, -1, 5 \rangle
\]

is parallel to \( C \). Therefore

\[
\mathbf{r}(t) = \left( \frac{2}{5}, -\frac{2}{5}, 0 \right) + t\langle -4, -1, 5 \rangle
\]

is a parametrization of \( C \).

More complicated curves are parametrized similarly. Typical points on a curve \( C \) are accessed by a position vector \( \mathbf{r} \) of the form

\[
\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle.
\]

For example, the parametrization \( \langle \sin t, \cos t, t \rangle \) describes a helix in \( R^3 \). Or the intersection of the plane \( x + y + z = 1 \) and the cylinder \( x^2 + y^2 = 1 \) is given by

\[
\mathbf{r}(t) = \langle \cos t, \sin t, 1 - \cos t - \sin t \rangle, \quad t \in (0, 2\pi].
\]

3 Parametrization of Surfaces

Surfaces in \( R^3 \) are characterized by two degrees of freedom; one is allowed to vary two parameters independently to cover all points on a surface. The simplest examples are surfaces that are graphs of functions \( f \) that depend on two variables, \( z = f(x, y) \). Such surfaces are often parametrized as

\[
\mathbf{r}(x, y) = \langle x, y, f(x, y) \rangle, \quad a < x < b, \quad c < y < d.
\]

For example, the surface \( z = x^2 + y^2 \) over the unit square is parametrized as

\[
\mathbf{r}(x, y) = \langle x, y, x^2 + y^2 \rangle, \quad 0 < x < 1, \quad 0 < y < 1.
\]

The cylinder \( x^2 + y^2 = 1 \) is parametrized as

\[
\mathbf{r}(\theta, z) = \langle \cos \theta, \sin \theta, z \rangle, \quad \theta \in (0, 2\pi], \quad z \in \mathbb{R},
\]

while the cylinder \( x^2 + z^2 = 4 \) is parametrized as

\[
\mathbf{r}(\theta, y) = \langle 2\cos \theta, y, 2\sin \theta \rangle, \quad \theta \in (0, 2\pi], \quad y \in \mathbb{R},
\]
The surface of the disk of radius $a$ in the plane $z = b$ centered at the origin is given by
\[ r(u, v) = \langle u \cos v, u \sin v, b \rangle, \quad u \in [0, 1], \quad v \in (0, 2\pi]. \] (21)

Certain surfaces are best parametrized in spherical coordinates where
\[
\begin{cases}
  x = \rho \cos \theta \sin \phi, \\
y = \rho \sin \theta \sin \phi, \\
z = \rho \cos \phi.
\end{cases}
\] (22)

For example, the cone $z^2 = x^2 + y^2$ can be parametrized as
\[ r(\rho, \theta) = \frac{\sqrt{2}}{2} (\rho \cos \theta, \rho \sin \theta, \rho), \quad \rho \in R, \quad \theta \in (0, 2\pi]. \] (23)

Similarly, the northern hemisphere of radius 3 centered at the origin may be parametrized as
\[ r(\theta, \phi) = 3 (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi), \quad \theta \in (0, 2\pi], \quad \phi \in [0, \frac{\pi}{2}]. \] (24)

An alternative way of parametrizing this surface is as follows:
\[ r(x, y) = 3 (x, y, \sqrt{9 - x^2 - y^2}), \quad x^2 + y^2 \leq 9. \] (25)

The boundary of this surface (the circle of radius 3 in the $xy$-plane and centered at the origin) is best parametrized using (24) by setting $\phi = \frac{\pi}{2}$ in that relation to get
\[ r(\theta) = 3 (\cos \theta, \sin \theta, 0), \quad \theta \in (0, 2\pi]. \] (26)

Once a parametrization $r(u, v)$ of a surface $S$ is known, the vector
\[ r_u \times r_v \]
defines a normal vector to $S$.

4 Parametrization of Regions in $R^3$

Regions in $R^3$ have three degrees of freedom. They are parametrized by $r(u, v, w)$ where $u$, $v$ and $w$ take on values in respective intervals. For example, the region bounded by the cylinder $x^2 + y^2 = 1$ and the planes $z = -2$ and $z = 1$ is parametrized as
\[ r(r, \theta, z) = \langle r \cos \theta, r \sin \theta, z \rangle, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi, \quad -2 \leq z \leq 1. \] (27)

The boundary of this region consists of three surfaces $S_1$, $S_2$ and $S_3$ given by
\[
\begin{cases}
  S_1 : \quad r_1(u, v) = \langle u \cos v, u \sin v, -2 \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v < 2\pi, \\
  S_2 : \quad r_2(u, v) = \langle u \cos v, u \sin v, 1 \rangle, \quad 0 \leq u \leq 1, \quad 0 \leq v < 2\pi, \\
  S_3 : \quad r_3(u, v) = \langle \cos v, \sin v, u \rangle, \quad -2 \leq u \leq 1, \quad 0 \leq v < 2\pi.
\end{cases}
\] (28)
Similarly, the region inside the northern hemisphere of radius 2 is parametrized as follows:

\[
\mathbf{r}(\rho, \theta, \phi) = \langle \rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi \rangle, \quad 0 \leq \rho \leq 2, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{2}
\]

(29)

The boundary of this region consists of two surfaces \( S_1 \) and \( S_2 \) given by

\[
\begin{align*}
\begin{cases}
S_1 : & \mathbf{r}_1(\theta, \phi) = 2 \langle \cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi \rangle, \quad 0 \leq \theta < 2\pi, \quad 0 \leq \phi < \frac{\pi}{2}, \\
S_2 : & \mathbf{r}_2(r, \theta) = \langle r \cos \theta, r \sin \theta, 0 \rangle, \quad 0 \leq r \leq 2, \quad 0 \leq \theta < 2\pi.
\end{cases}
\end{align*}
\]

(30)