Torsion Divisor Classes

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Outline

Background

Results

Example
Let $A \to B$ be a homomorphism of Noetherian normal integral domains. Under certain circumstances, one can show that such a map induces a group homomorphism $\text{Cl}(A) \to \text{Cl}(B)$.

- (faithfully) flat ring homomorphisms; The map $\text{Cl}(A) \to \text{Cl}(B)$ is given by $[a] \mapsto [a \otimes_A B]$.
- integral extensions;
- (Danilov, 1970) the natural surjection $A[[T]] \to A$;
- (Lipman, 1979) any surjection of the form $A \to A/tA$, assuming that $A$ and $A/tA$ are both Noetherian normal integral domains.
• (S-W, S, 2009) homomorphisms of finite flat dimension between normal domains.

  e.g., $A \rightarrow A/I$, where $I$ is a prime ideal of $A$ with finite projective dimension, $A$ and $A/I$ are normal domains. The map $\text{Cl}(A) \rightarrow \text{Cl}(B)$ is given by $[a] \mapsto [(a \otimes_A B)^{BB}]$.

• (S-W, S, 2013) $A \rightarrow (A/I)'$, where $A$ is a normal domain, $I$ a prime ideal of $A$ such that $A/I$ is excellent and satisfies $(R_1)$, and $(A/I)'$ the integral closure of $A/I$.

The kernel of this induced map has been studied by V.I. Danilov, P. Griffith, J. Lipman, C. Miller, D. Weston, and others.
Some Definitions

The **dual** of an $A$-module $a$ is $a^A := \text{Hom}_A(a, A)$.

There is a map $\sigma : a \rightarrow a^{AA}$, defined by $\sigma(x) = \varphi_x$, where $\varphi_x(f) := f(x)$, for $f \in \text{Hom}_A(a, A)$. If $\sigma$ is an isomorphism, then $a$ is said to be **reflexive**.

The **divisor class group** of $A$, denoted $\text{Cl}(A)$, is the group of isomorphism classes of reflexive ideals of $A$, or equivalently, reflexive $A$-modules of rank one. An element $[a]$ in $\text{Cl}(A)$ is called a **divisor class**.

- multiplication is defined by $[a] \cdot [b] = [(a \otimes_A b)^{AA}]$;
- the identity element is $[A]$;
- $[a]^{-1} = [a^A] = \text{Hom}_A(a, A)$.
FACT: A Noetherian integral domain is a unique factorization domain if and only if every height one prime is principal.

Example

- Let $A = \mathbb{C}[X, Y, Z, W]/(XY - ZW)$. Then $\text{Cl}(A) \cong \mathbb{Z}$.
- Let $B = \mathbb{C}[X, Y, Z]/(XY - Z^2)$. Then $\text{Cl}(B) \cong \mathbb{Z}/2\mathbb{Z}$.

There is a map $\text{Cl}(A) \to \text{Cl}(B)$ induced from $A \to B$. 
Theorem (Griffith, Weston 1994*)

Let \((A, \mathfrak{m})\) be an excellent, local, normal domain and let \(t\) be a principal prime element in \(\mathfrak{m}\) such that \(A/tA\) satisfies the condition \((R_1)\). Let \(e > 1\) be an integer which represents a unit in \(A\). Then the kernel of the homomorphism \(\text{Cl}(A) \to \text{Cl}(\langle A/tA \rangle')\) contains no element of order \(e\).

Recall: The map \(\text{Cl}(A) \to \text{Cl}(B)\) is given by \([a] \mapsto [(a \otimes_A B)^{BB}]\), where \(B = (A/I)'\).

Outline of their proof

- Suppose \([a]\) is an element of order \(e\) in kernel of \(\text{Cl}(A) \rightarrow \text{Cl}((A/tA)')\); let \(\alpha \in A\) be such that \(a^e = \alpha A\).

- They describe an \(A\)-algebra structure on the direct sum \(R = A \oplus a \oplus a^{(2)} \oplus \cdots \oplus a^{(e-1)}\) that makes \(R\) isomorphic to the integral closure of \(A[T]/(T^e - \alpha)\).

- They then show that the integral closure of \(R \otimes_A (A/tA)'\) is étale over \((A/tA)'\) and conclude that \(e = 1\).

- **Subtle Detail:** That the integral closure of \(R \otimes_A (A/tA)'\) is local.
The best way to be certain of this claim is to use a result of M. Hochster and C. Huneke, interpreting the integral closure of $R \otimes_A (A/tA)'$ as the $S_2$-ification of $R \otimes_A (A/tA)'$.


**Fact (Hochster, Huneke)**

Let $(B, \mathfrak{n})$ be an excellent equidimensional local ring. If $B$ is $(S_2)$ and $x_1, \ldots, x_k$ is a part of a system of parameters, then the $S_2$-ification of $B/(x_1, \ldots, x_k)B$ is local.
**Definition**

For a Noetherian integral domain $B$, a subring $S$ of the total ring of quotients of $B$ is an $S_2$-ification of $B$ if:

- $S$ is module finite over $B$;
- $S$ satisfies the Serre condition ($S_2$) over $B$; and
- $\text{Coker}(B \to S)$ has no prime ideal of $B$ of height less than two in its support.

Recall: $S$ satisfies the ($S_2$) condition over $B$ if for every prime ideal $p$ of $B$, $\text{depth } S_p \geq \min\{2, \text{ht } p\}$.

**FACT:** If $A$ is a local Noetherian normal domain that satisfies ($R_1$), then the $S_2$-ification $S$ of $A$ coincides with the integral closure of $A$ in its field of fractions.
Theorem One [Sather-Wagstaff, S-]
Let \((A, m)\) be an excellent, local, normal domain and \(I\) a prime ideal of \(A\) with finite projective dimension such that:

(i) \(\overline{A} = A/I\) is normal;

(ii) \(\mu(I) \leq \dim A - 2\), where \(\mu(I)\) is the minimal number of generators of \(I\); and

(iii) \(I\) is a complete intersection on the punctured spectrum of \(A\), i.e., for each prime ideal \(p \neq m\), the localization \(I_p\) is either equal to \(A_p\) or generated by a regular sequence in \(A_p\).

Then for any integer \(e > 1\) which represents a unit in \(A\), the kernel of the homomorphism \(\text{Cl}(A) \to \text{Cl}(\overline{A})\) contains no elements of order \(e\).
Fact (Hochster, Huneke 1992**)

For \((T, \mathfrak{n})\) a complete local equidimensional ring such that \((0)\) has no embedded primes, the following conditions are equivalent:

- \(H^\dim_n T (T)\) is indecomposable;
- The canonical module of \(T\) is indecomposable;
- The \(S_2\)-ification of \(T\) is local;
- For every ideal \(J\) of height at least two, \(\text{Spec}(T) - V(J)\) is connected;
- Given any two distinct minimal primes \(p, q\) of \(T\), there is a sequence of minimal primes \(p = p_0, \ldots, p_r = q\) such that for \(0 \leq i < r\), \(\text{ht}(p_i + p_{i+1}) \leq 1\).
A significant tool in the proof of theorem one is the next result.

**Theorem Two** [Sather-Wagstaff, S-] Let \((A, \mathfrak{m})\) be a complete, local, normal domain such that \(A/\mathfrak{m}\) is separably closed and let \(I\) be a prime ideal of \(A\) with finite projective dimension such that \(\overline{A} = A/I\) is normal. Let \(e > 1\) be an integer which represents a unit in \(A\) and assume that \(A\) contains a primitive \(e\)-th root of unity. Let \([a]\) be an element in \(\text{Cl}(A)\) with order \(e\). Set \(R = A \oplus a \oplus a^{(2)} \oplus \cdots \oplus a^{(e-1)}\). If any of the five equivalent conditions of the HH Fact hold for \(R/IR\), then \([a] \notin \text{Ker}(\text{Cl}(A) \rightarrow \text{Cl}(\overline{A}))\).
Lemma: Let $P \in \text{Spec}(R)$ and set $p = P \cap A$. Then $\text{ht } p = \text{ht } P$. Furthermore, if $A_p$ is regular, then the extensions $A_p \to R_p$ and $A_p \to R_P$ are étale. In particular, the extension $A \to R$ is étale in codimension one.

Definition
We say that $R$ is étale in codimension one (or more accurately, in codimension less than or equal to one) over $A$ if, for every prime ideal $P$ of $S$ with height less than or equal to one and $p = P \cap A$, the ring $R_P$ is étale over $A_p$. 
**Lemma:** The ring $R$ is a complete local normal domain.

**Lemma:** Set $\bar{R} = R \otimes_A \bar{A} = R/IR$. The ring $\bar{R}$ is equidimensional, complete, and local. In particular, Fact HH applies with $T = \bar{R}$.

**Lemma:** The extension $\bar{A} \to \bar{R}$ is étale in codimension one. Moreover, the ring $\bar{R}$ satisfies the Serre condition ($R_1$).
**Lemma:** For $\sigma : \bar{R} \to \bar{R}^{\bar{A}\bar{A}}$, we have $\text{Ker} \sigma = j(\bar{R})$.

**Definition**
Denote by $j(\bar{R})$ the largest ideal which is a submodule of $\bar{R}$ of dimension smaller than $\dim \bar{R}$. Specifically,

$$j(\bar{R}) = \{\bar{r} \in \bar{R} : \dim(\bar{R}/\text{ann}_{\bar{R}}(\bar{r})) < \dim \bar{R}\}.$$  

**Lemma:** The ring $\bar{R}/j(\bar{R})$ satisfies $(R_1)$. Its integral closure $(\bar{R}/j(\bar{R}))'$ in its total ring of quotients is its $S_2$-ification.
Lemma: The $\bar{A}$-module $\bar{R}^{\bar{A}\bar{A}}$ is free of rank $e$ and thus satisfies $(S_2)$ over $\bar{A}$.

Lemma: $(\bar{R}/j(\bar{R}))'$ satisfies $(S_2)$ as an $\bar{A}$-module and as a ring.

Lemma: $(\bar{R}/j(\bar{R}))' \cong (\bar{R}/j(\bar{R}))^{\bar{A}\bar{A}} \cong \bar{R}^{\bar{A}\bar{A}}$ as $\bar{A}$-isomorphisms.

Remark
Upshot: $(\bar{R}/j(\bar{R}))'$ is the $S_2$-ification of $\bar{R}$. 

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Lemma: The ring $(\bar{R}/j(\bar{R}))'$ is a normal domain.

Lemma: The map $\bar{A} \rightarrow (\bar{R}/j(\bar{R}))'$ is étale in codimension one.

Lemma: The extension $\bar{A} \hookrightarrow (\bar{R}/j(\bar{R}))'$ is étale.
Conclusion of proof of Theorem Two.

Let $S$ denote $(\bar{R}/j(\bar{R}))'$. The extension $\bar{A} \hookrightarrow S$ is étale and local and $S$ is free over $\bar{A}$ of rank $e$. There are isomorphisms

$$\bar{A}/\bar{m} \cong S/\bar{m}S \cong (\bar{A}/\bar{m})^e.$$ 

It follows that $e = 1$. Contradiction.
Theorem One Hypotheses:

$(A, \mathfrak{m})$ is an excellent, local, normal domain and $I$ a prime ideal of $A$ with finite projective dimension such that:

(i) $\overline{A} = A/I$ is normal;

(ii) $\mu(I) \leq \dim A - 2$, where $\mu(I)$ is the minimal number of generators of $I$; and

(iii) $I$ is a complete intersection on the punctured spectrum of $A$, i.e., for each prime ideal $\mathfrak{p} \neq \mathfrak{m}$, the localization $I_\mathfrak{p}$ is either equal to $A_\mathfrak{p}$ or generated by a regular sequence in $A_\mathfrak{p}$. 
Example
Let \((A, m)\) be an excellent normal local integral domain, and let the positive integer \(e\) represent a unit in \(A\). Assume that \(A\) has an \(A\)-regular sequence \(f_1, \ldots, f_6 \in m\) such that \(A/(f_1, \ldots, f_6)A\) is a normal domain. (Examples of such rings are constructed in S-W, S 2009.) In particular, we have \(6 \leq \text{depth} \ A \leq \text{dim} \ A\).

Localize at a minimal prime of the ideal \(J = (f_1, \ldots, f_6)A\), if necessary, to assume that \(J\) is \(m\)-primary. It follows that \(A\) is Cohen-Macaulay of dimension 6. Arrange the sequence \(f_1, \ldots, f_6\) in a \(2 \times 3\) matrix \(F = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \end{pmatrix}\), and consider the ideal \(I = I_2(F)\) generated by the \(2 \times 2\) minors \(d_1 = f_1f_5 - f_2f_4\), \(d_2 = f_2f_6 - f_3f_5\), \(d_3 = f_1f_6 - f_3f_4\).
Thank You, eh.

Merci, eh.
Let $A$ be a ring, and let $R$ be an $A$-algebra.

**Definition**
For a prime ideal $P$ of $R$ and $p = P \cap A$, we say that $P$ is *unramified over* $A$ if $pR_P = PR_P$ and $R_P/pR_P$ is a separable field extension of $A_p/pA_p$.

**Definition**
We say that $R$ is *unramified in codimension* $i$ (or more accurately, in codimension less than or equal to $i$) over $A$ if every prime ideal of $R$ with height less than or equal to $i$ is unramified over $A$.

**Definition**
For $R$ to be *unramified over* $A$ means that (1) every prime ideal of $R$ is unramified over $A$, and (2) for every $p \in \text{Spec}(A)$ there are only finitely many prime ideals of $R$ lying over $p$. 
• $A$ satisfies the hypotheses of the first Theorem (excellent, local, normal domain);

• $I$ is a height-2 prime ideal of $A$ of finite projective dimension generated by 3 elements (so $\mu(I) < \dim A - 2$) such that $A/I$ is a normal domain;

• $I$ is a complete intersection on the punctured spectrum; i.e., for $p \in \text{Spec}^\circ(A)$ such that $I \subseteq p$, $I_p$ is generated by an $A_p$-regular sequence.
\[ a = p_1^{(n_1)} \cap \cdots \cap p_r^{(n_r)}, \text{ } p_i \text{ height one primes in } A \]

\[ a^{(e)} = p_1^{(en_1)} \cap \cdots \cap p_r^{(en_r)} = \alpha A \text{ and } v_{p_i}(a) = en_i \]

\[ \exists b \in A \text{ such that } v_{p_i}(b) = n_i \text{ for } i = 1, \ldots, r, \text{ else } v_q(b) \geq 0 \]

\[ \text{Set } u = b^e/a \in K. \text{ Then } v_{p_i}(u) = en_i - en_i = 0 \text{ and } v_q(u) = v_q(b^e) \geq 0. \]

Let \( \sqrt[e]{u} \) be a fixed \( e \)-th root of \( u \) in an algebraic closure of \( K \). Then \( \sqrt[e]{a} := b/\sqrt[e]{u} \) is an \( e \)-th root of \( a \). Also, \( au = b^e \).
The ring

\[ R = A \oplus \left[ a \cdot \frac{\sqrt[e]{u}}{b} \right] \oplus \left[ a^2 \cdot \frac{\sqrt[e]{u^2}}{b^2} \right] \oplus \cdots \oplus \left[ a^{e-1} \cdot \frac{\sqrt[e]{u^{e-1}}}{b^{e-1}} \right] \]

is the integral closure of \( A \) in the field extension \( K[\sqrt[e]{a}] = K[\sqrt[e]{u}] \); i.e., \( R \) is a domain.

The ring structure on \( R \): for \( a_i \in a^{(i)} \) and \( a_j \in a^{(j)} \), we have

\[
\left( a_i \frac{\sqrt[e]{u^i}}{b^i} \right) \left( a_j \frac{\sqrt[e]{u^j}}{b^j} \right) = \begin{cases} 
  a_i a_j \frac{\sqrt[e]{u^{i+j}}}{b^{i+j}} & \text{if } i + j < e \\
  a_i a_j \frac{\sqrt[e]{u^{i+j}}}{b^{i+j}} = \frac{a_i a_j \sqrt[e]{u^{i+j-e}}}{a \frac{b^{i+j-e}}{b^{i+j-e}}} & \text{if } i + j \geq e.
\end{cases}
\]
Key References


Proposition

Let $C \hookrightarrow D$ be a module finite extension of local normal domains which is unramified in codimension one. If $t$ is an element in the maximal ideal of $C$ such that $\bar{C} = C/tC$ satisfies the condition $(R_1)$, then $\bar{D} = D/tD$ satisfies the condition $(R_1)$ as well, and $\bar{C} \hookrightarrow \bar{D}$ is unramified in codimension one.
Corollary

Under the same hypotheses as in the above Proposition, if the rings are complete, then the integral closures $(\bar{C})'$ and $(\bar{D})'$ are local integral domains.

Corollary

Under the same hypotheses as in the previous corollary, the extension $(\bar{C})' \to (\bar{D})'$ is module finite and unramified in codimension one.