Local cohomology at determinantal ideals

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Notation

- $R = \mathbb{Z}[X]$, $X = m \times n$ matrix of indeterminates.
- $R_p = R/pR$; $R_0 = \mathbb{Q} \otimes_{\mathbb{Z}} R$.
- $I_t =$ideal of $t$-minors of $X$.

Object of desire:

$H^\bullet_{I_t}(R)$, local cohomology modules.
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Local cohomology

\[ A = \text{Noetherian ring, } \alpha = A\text{-ideal}, \]
\[ f_1, \ldots, f_k \in \alpha \text{ with } \sqrt{\alpha} = \sqrt{f_1, \ldots, f_k}. \]

- \( C_f^s = \bigoplus A[1/(f_{i_1} \cdots f_{i_s})]. \)
- \( C_f^s \rightarrow C_f^{s+1} \) sum of signed localization maps
- Gives complex \( C_f^\bullet \) such that
  - cohomology is supported inside \( \alpha \),
  - different choices of \( f \) give homotopic complexes,
  - even if they generate different ideals.

**Notation:** \( H^s(C_f^\bullet) =: H_\alpha^s(A) \) for any choice \( f \).
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- $C_f^s = \bigoplus A[1/(f_1 \cdots f_s)]$.
- $C_f^s \longrightarrow C_f^{s+1}$ sum of signed localization maps
- Gives complex $C_f^\bullet$ such that
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**Notation:** $H^s(C_f^\bullet) =: H^s_{\mathfrak{a}}(A)$ for any choice $f$. 
**Local cohomology, II**

Some properties:

- Local cohomology vanishes beyond arithmetic rank.
- Alternatively: $H^s_a(\cdot) = \mathbb{R}^s(\Gamma_a(\cdot))$ where
  \[
  \Gamma_a(M) = \{ m \in M \mid \text{supp}(A \cdot m) \subseteq \text{Var}(a) \}.
  \]

- It is the algebraic geometer’s version of relative cohomology:
  - $Y \subseteq X$ closed; $\mathcal{F}$ sheaf on $X$, then get exact natural
    \[
    H^s_Y(X, \mathcal{F}) \longrightarrow H^s(X, \mathcal{F}) \longrightarrow H^s(X \setminus Y, \mathcal{F}) \longrightarrow H^{s+1}_Y(X, \mathcal{F})
    \]
- $H^s_a(A) = \lim_{r \to} H^s(A, \{ f_1^r, \ldots, f_k^r \})$.

- related to:
  - topology if $\mathbb{Q} \subseteq A$;
  - tight closure if $\mathbb{Z}/p\mathbb{Z} \subseteq A$.

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Some methods

- $H^s_\alpha(A)$ is usually not Noetherian.
- Frivolous example: $A = \mathbb{k}[x]$, $\alpha = (x)$. Then $H^s_\alpha(A) = 0$ for $s \neq 1$ while $H^1_\alpha(A)$ is
  \[
  \frac{\mathbb{k}\langle x, \partial_x \rangle \cdot \frac{1}{x}}{A}
  \]
  if $\mathbb{Q} \subseteq A$,
  \[
  \bigcup_e (A/F^e(\alpha))
  \]
  if $\mathbb{Z}/p\mathbb{Z} \subseteq A$,
  and equal to $\bigoplus_{r<0} \mathbb{k} \cdot x^r$ in either case.
- Lessons from the example:
  - $H^s_\alpha(A)$ is module over $A[F]$ if $\mathbb{Z}/p\mathbb{Z} \subseteq A$,
  - $H^s_\alpha(A)$ is module over $D(A, \mathbb{Q})$ if $\mathbb{Q} \subseteq A$,
  - $H^s_\alpha(A)$ is module over $D(A, \mathbb{Z})$ in any case.

These rings are bigger, not commutative, not always Noetherian.
Some methods

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In any characteristic: (but over a domain)
- $I_t$ is prime \((\text{de Concini–Eisenbud–Procesi})\)
- Cohen–Macaulay \((\text{Hochster–Eagon})\)
- of height $h_t := (mt + 1)(nt + 1)$
- and arithmetic rank $mn - t^2 + 1$. \((\text{Bruns–Schwänzl})\)

In positive characteristic,
- Frobenius powers of $I_t$ are also perfect.
- So $H_{I_t}^{> h_t}(R_p) = 0$.

In characteristic zero,
- Powers of $I_t$ or $(f_1^r, \ldots, f_k^r)$ not CM, even for $m = 2 = n - 1 = t$.
- $H_{I_t}^{> h_t}(R)$ not necessarily zero.
Motivation:

- If \( m = 2 = n - 1 = t \), ("the 2 \times 3 case") then \( h_2 = 2 \), \( \text{ara}(l_2) = 3 \).
- \( H^3_{l_2}(R_0) \neq 0 \) (Hochster, -).
- Huneke, Katz, Marley: if \( \phi: R_0 \rightarrow A \supset \mathbb{Q} \) with \( \dim(\phi(R_0)) < 6 \) then \( H^3_{l_2}(A) = 0 \)
- Does not follow from general vanishing theorems à la Grothendieck–Faltings–Huneke–Lyubeznik.

Back to \( R \):

- there is an exact sequence

\[
0 \rightarrow \mathbb{Z}\text{-torsion} \hookrightarrow H^3_{l_2}(R) \xrightarrow{\iota} H^3_{l_2}(R_0) \rightarrow \mathbb{Z}\text{-torsion} \rightarrow 0
\]

- Singh proved: \( H^3_{l_2}(R) \) is \( \mathbb{Q} \)-space, so \( \iota \) iso!
Motivation:

- If $m = 2 = n - 1 = t$, ("the $2 \times 3$ case") then $h_2 = 2$, $\text{ara}(l_2) = 3$.
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- Singh proved: $H^3_{l_2}(R)$ is $\mathbb{Q}$-space, so $\iota$ iso!
Main results:

Theorem

\[ R = \mathbb{Z}[X], \ l_t \text{ as above.} \]

1. \( H^k_{l_t}(R) \) is \( \mathbb{Z} \)-torsion-free for all \( t, k \).
2. If \( k \neq h_t \), then \( H^k_{l_t}(R) \) is a \( \mathbb{Q} \)-space, iso to \( H^k_{l_t}(R_0) \).
3. Consider the \( \mathbb{N} \)-grading on \( R \) with \([R]_0 = \mathbb{Z} \) and \( \deg x_i = 1 \). If \( 2 \leq t \leq \min(m, n) \) and \( t < \max(m, n) \) then

\[
H^{mn-t^2+1}_{l_t}(\mathbb{Z}[X]) \cong E_{R_0}(\mathbb{Q})(mn) \quad \text{(this is a shift!)}
\]
As a consequence, one obtains:

**Theorem**

Let $a = t$-minors of $M \in A^{m \times n}$, $A$ Noetherian, $1 \leq t \leq \min(m, n)$, $t < \max(m, n)$.

If $\dim A < mn$ then $H^{mn-t^2+1}_a(A) = 0$. 
- **Frobenius functor** $F : (\_ \rightarrow A') \otimes_A (\_ \rightarrow)$
  $A'$ left $A$-module as expected; the right action $a'a = a^p a'$.

- **$F$-module** is a direct limit

\[
M \rightarrow F(M) \rightarrow F(F(M)) \rightarrow \cdots \rightarrow M
\]

for some $A$-module $M$ and some $A$-morphism $M \rightarrow F(M)$.

- **$F$-finite** if $M$ is Noetherian.

- **$A\{f\}$-module** is a module over $A\{f\} = A\langle f \rangle / \{r^p f - fr\}$.

  - $F$-finite modules are finite length as $F$-modules. (Lyubeznik)
  - $H^r_x(\mathbb{k}[x]) \cong E_{\mathbb{k}[x]}(\mathbb{k})$ not injective as $F$-module (Ma)
Main technical results in char \( p > 0 \)

**Theorem**

\[ R_p = (\mathbb{Z}/p\mathbb{Z} \subseteq \mathbb{k})[x]; \; m = (x), \; I \text{ graded.} \]

\( \forall k \geq 0, \) TFAE:

1. The \( D(R_p, \mathbb{k}) \)-module \( H^k_I(\mathcal{R}_p) \) has a composition factor with support \( \{m\} \).
2. The graded \( F \)-finite module \( H^k_I(\mathcal{R}_p) \) has a composition factor with support \( \{m\} \).
3. \( H^k_I(\mathcal{R}_p) \) has a graded \( F \)-module quotient with support \( \{m\} \).
4. The natural Frobenius action on \( [H^*_{m, \dim(A)}(\mathcal{R}_p/I)]_0 \) is not nilpotent.

**Theorem**

\( H^r_x(\mathbb{k}[x]) \) injective as *graded* \( F \)-module.