Plan

- Invariant Theory
  - Rings of invariants, $R^G$
  - Derksen’s algorithm

- Differential symmetries of $R^G$
  - Ideals in the Weyl algebra
  - The symmetry algebra and $D(R^G)$
  - Computing $D(R^G)$
Rings of Invariants

When a group $G$ acts on $X = \text{Spec}(R)$, it is customary to consider the categorical quotient $X//G = \text{Spec}(R^G)$.

The ring $R^G$ is the ring of invariants:

$$R^G = \{ r \in R : g \cdot r = r \text{ for all } g \in G \}.$$

**Ex:** If $G = \langle \sigma : \sigma^2 = \text{id} \rangle$ acts on $R = \mathbb{C}[x,y]$ via $\sigma \cdot x = -x$ and $\sigma \cdot y = -y$ then $R^G = \mathbb{C}[x^2, xy, y^2]$. 
Example: binary forms

The geometry of (pairs of) points on $\mathbb{P}^1$ is controlled by the (deg 2) forms.

Let $X = \{ax^2 + bxy + cy^2\} = \text{Proj } \mathbb{C}[a,b,c]$.

$G = \text{SL}_2 \mathbb{C}$ acts on $\mathbb{P}^1$ and moves the roots of a form, so $G$ acts on $X$:

- $g \cdot a = g_{1,1}^2 a + 2g_{1,1}g_{2,1} b + g_{2,1}^2 c$
- $g \cdot b = g_{1,1}g_{1,2} a + (g_{1,1}g_{2,2} + g_{1,2}g_{2,1}) b + g_{2,1}g_{2,2} c$
- $g \cdot c = g_{1,2}^2 a + 2g_{1,2}g_{2,2} b + g_{2,2}^2 c$

$R^G = \mathbb{C}[a,b,c]^{\text{SL}_2 \mathbb{C}} = \mathbb{C}[b^2-4ac]$
Properties of $R^G$

It is generally difficult to compute $R^G$ explicitly (c.f. Takashi Wada’s talk).

Gordan and Hilbert showed that $R^G$ is **finitely generated** when $G$ is lin. reductive and Nagata gave a counterexample when $G$ is not lin. reductive.
Computing invariants

Several methods:
(1) Gordan’s symbolic calculus (P. Olver)
(2) Cayley’s omega process
(3) Lie algebra methods (Sturmfels)
(4) Derksen’s algorithm

Harm Derksen

Gregor Kemper
Derksen’s Algorithm

(1) **Hilbert ideal** \( I = \text{ideal of } R \text{ gen by } R^G_{>0} \)

(2) To find \( I \), we first look at the map

\[
\psi : G \times X \rightarrow X \times X \quad \text{B} = \text{im}(\psi)
\]

\[
(g, x) \mapsto (x, g \cdot x) \quad \beta = \text{ideal}(B)
\]

Hilbert - Mumford Criterion:

\[
B \cap (X \times \{0\}) = V(I) \times \{0\}
\]

\[
\beta + (y_1, \ldots, y_n) = I + (y_1, \ldots, y_n)
\]

Compute the ideal \( \beta \) by **elimination** and set \( y \)'s to 0 to get gens for the Hilbert ideal.

(3) The gens of \( I \) may not be invariants but we can average them over the group action to get invariants that generate \( I \) and \( R^G \).
Example: binary forms

To describe the map $\psi : G \times X \to X \times X$
we need to represent $G = \text{SL}_2 \mathbb{C}$ as a variety:

$G = V(g_{1,1}g_{2,2} - g_{2,1}g_{1,2} - 1) \subset \mathbb{C}[g_{1,1}, g_{1,2}, g_{2,1}, g_{2,2}]$. 
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Differential symmetries

One way to study any ring of functions is to understand the ring as the solutions to a system of differential equations.

Cayley developed a system of diff. eqs that characterize the invariants of forms (the G=SL\(_2\)\(\mathbb{C}\) case).

Arthur Cayley
The Lie algebra method works with $\text{GL}_2\mathbb{C}$ rather than $\text{SL}_2\mathbb{C}$. There is a close connection between the invariants of these two groups.

A function $f$ is said to be an “invariant of index $\gamma$” if $g \cdot f = (\det g)^\gamma f$ for all $g \in \text{GL}_2\mathbb{C}$.

**Theorem:** For binary forms, \( \text{GL}_2\mathbb{C} \)-invariants of index $\gamma = \text{homog.} \ \text{SL}_2\mathbb{C} \)-invariants of degree $2\gamma/d$. 
Example: degree 2 forms

Degree 2 form: \( F(x,y) = ax^2 + bxy + cy^2 \).

\[

g \cdot a = g_{1,1}^2 a + 2g_{1,1}g_{2,1} b + g_{2,1}^2 c \\
g \cdot b = g_{1,1}g_{1,2} a + (g_{1,1}g_{2,2} + g_{1,2}g_{2,1}) b + g_{2,1}g_{2,2} c \\
g \cdot c = g_{1,2}^2 a + 2g_{1,2}g_{2,2} b + g_{2,2}^2 c \\
\]

If \( g = \begin{bmatrix} \lambda & 0 \\ 0 & \tau \end{bmatrix} \): \( g \cdot a = \lambda^2 a, \quad g \cdot b = \lambda \tau b, \quad g \cdot c = \tau^2 c \)

Now if \( f(a,b,c) \) has index \( \gamma \) then

\( f(\lambda^2 a, \lambda \tau b, \tau^2 c) = (\lambda \tau)^\gamma f(a,b,c) \). This forces

\[
(2a \partial_a + 1b \partial_b + 0c \partial_c )f = \gamma f \\
(0a \partial_a + 1b \partial_b + 2c \partial_c )f = \gamma f
\]
Example continued

\[(2a\partial_a + 1b\partial_b + 0c\partial_c)f = \gamma f\]

\[(0a\partial_a + 1b\partial_b + 2c\partial_c)f = \gamma f\]

There are two other types of matrices that gen. GL\(_2\mathbb{C}\):

\[
\begin{bmatrix}
1 & * \\
0 & 1
\end{bmatrix}
gives the condition \((1a\partial_b + 2b\partial_c)f = 0\) and

\[
\begin{bmatrix}
1 & 0 \\
* & 1
\end{bmatrix}
gives the condition \((2b\partial_a + 1c\partial_b)f = 0\).
Cayley’s Differential System

\[(2a\partial_a + 1b\partial_b + 0c\partial_c)f = \gamma f\]
\[(0a\partial_a + 1b\partial_b + 2c\partial_c)f = \gamma f\]
\[(1a\partial_b + 2b\partial_c)f = 0\]
\[(2b\partial_a + 1c\partial_b)f = 0\]

\[ax^2 + bxy + cy^2 \quad \longrightarrow \quad a_2x^2 + a_1xy + a_0y^2\]

\[\longrightarrow \quad \sum a_i x^i y^{d-i}\]
Cayley’s Differential System

\[(\sum ia_i \partial_i - \gamma)f = 0\]

\[(\sum (d - 2i)a_i \partial_i)f = 0\]

\[(\sum (d - i)a_{i+1} \partial_i)f = 0\]

\[(\sum ia_{i-1} \partial_i)f = 0\]

\[ax^2 + bxy + cy^2 \longrightarrow a_2x^2 + a_1xy + a_0y^2\]

\[\longrightarrow \sum a_ix^iy^{d-i}\]
Ideals in the Weyl algebra

We can make Cayley’s system into an algebraic object using the Weyl algebra.

**Weyl algebra:** $W = \mathbb{C}[x_1, \ldots, x_n]<\partial_1, \ldots, \partial_n>$

Systems of linear PDEs $\iff$ left ideals in $W$

- $f$ satisfies $\Delta_1 f = \Delta_2 f = \ldots = 0$ $\iff$ $f$ is annihilated by the left ideal $W(\Delta_1, \Delta_2, \ldots)$

**Solutions** to system of DEs correspond to elements of $\text{Hom}_W(W/J, \mathbb{C}[x_1, \ldots, x_n])$
Cayley’s Differential System

\[ (\sum ia_i \partial_i - \gamma)f = 0 \]

\[ (\sum (d - 2i)a_i \partial_i)f = 0 \]

\[ (\sum (d - i)a_{i+1} \partial_i)f = 0 \]

\[ (\sum ia_{i-1} \partial_i)f = 0 \]

\( J_d \) : characterizes invariants

\( J_d(\gamma) \) : invariants of order \( \gamma \)
M. Saito and I used the symmetry algebra to study systems of hypergeometric DEs.

\[
S(W/J) = \left\{ \Delta \in W : J\Delta \subseteq J \right\}
\]

A: matrix, \( \beta \): complex vector \( \rightarrow H_A(\beta) \)
\( \Delta \in S(W/J_A) \)

Then \( f \) is a soln to \( H_A(\beta) \) \( \iff \)
\( \Delta \cdot f \) is a soln to \( H_A(\beta') \) for some \( \beta' \).
Symmetry algebra

\[ S(W/J) = \left\{ \Delta \in W : \Delta J \subseteq J \right\} J \]

Theorem: \[ S(W/J_d) = \left\{ \Delta \in W : \Delta \cdot R^G \subseteq R^G \right\} \subseteq D(R^G) \]

Equality holds when \( J_d = \text{Ann}_W(R^G) \).

**Question:** How can we compute \( S(W/J_d) \)?

This looks hard so we pass to the study of \( D(R^G) \).
Differential Operators

If R is a k-algebra then $D(R) \subset \text{End}_k R$.

In general $D(R)$ is very difficult to compute and may not be finitely generated.

One nice example: $D(C[x_1, \ldots, x_n]) = W$.

$G$ acts on $C[x_1, \ldots, x_n]$ via the matrix $M$ and this induces an action on the $\partial_i$'s via the matrix $(M^{-1})^T$.

So it makes sense to ask for $D(R)^G = W^G$, the ring of invariant differential operators.
Unfortunately, $D(R)^G$ is not equal to $D(R^G)$.

$\pi: R^G \to R$ induces $\pi^*: D(R)^G \to D(R^G)$ by restriction

$$\theta \in D(R)^G \quad \text{Reynolds}$$

Theorem: if $J_d = \text{Ann}_WR^G$ then

$\text{im}(\pi^*) \subset S(W/J_d) \subset D(R^G)$
Failure of surjectivity

We’ve got a map $\pi^*: D(R)^G \to D(R^G)$. 

Have $\text{Im}(\pi^*) \subset S(D(R)/J) \subset D(R^G)$. 

Musson and Van den Bergh showed that the map $\pi^*$ may not be surjective ($G = \text{torus}$). 

Ian Musson 

M. Van den Bergh
Surjective when it counts

Schwarz showed that the map $\pi^*$ is surjective in many cases of interest. In fact, he showed that the Levasseur-Stafford Alternative holds for $\text{SL}_2\mathbb{C}$ representations:

Either (1) $R^G$ is regular or
(2) the map $\pi^*$ is graded surjective

In most cases $R^G$ is not regular and so $\pi^*$ is (graded) surjective. Then: $\text{im}(\pi^*) = S(W/J_d) = D(R^G)$. 

Gerry Schwarz
Graded Surjectivity

The Weyl algebra $D(R) = W$ is filtered by order (degree in the $\partial$'s). The associated graded ring is just a polynomial ring in $2n$ variables.

Saying that the map $\pi^*: D(R)^G \rightarrow D(R^G)$ is graded surjective means that the induced map $[\text{gr}D(R)^G] \rightarrow \text{gr}D(R^G)$ is surjective.

So to produce generators of $\text{gr}D(R^G)$ (and hence $D(R^G)$) it is enough to get generators of $\text{gr}D(R)^G$. 
Computing $D(R^G)$

Since $\text{gr}D(R)^G = [\text{gr}D(R)]^G$ is a poly. ring, we can use Derksen’s algorithm to compute $[\text{gr}D(R)]^G$. These generators lift immediately to generators of $D(R^G)$.

Ex. $D(C[a_0,a_1,a_2]^{SL_2C}) = C[a_1^2-4a_0a_2, a_0\partial_0+a_1\partial_1+a_2\partial_2, \partial_1^2-\partial_0\partial_2]$

Bad news: $d>2$ seems hard!

Ex. $D(C[x,y]^{Z/2Z}) = C[x^2,xy,y^2,x\partial_x,x\partial_y,y\partial_x, y\partial_y, \partial_x^2, \partial_x\partial_y, \partial_y^2]$
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