Higher Derivations in Algebraic Geometry

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Thank You.
Overview

- Invariant Theory of Graphs,
- Steenrod Algebra,
- Algebraic Geometry,
- Higher Derivations,
- Jet Spaces and Applications.
Question: Can we find invariants to distinguish graphs up to isomorphism?

Graphs(n): set of labeled graphs on n vertices.

Can we find a set of functions $f_1, \ldots, f_k$ that distinguish isomorphism classes of Graphs(n)? By this we mean:

- all functions $f_i$ are constant on isomorphism classes
- if $\Gamma_1$ and $\Gamma_2$ are nonisomorphic graphs then there is a function $f_i$ with $f_i(\Gamma_1) \neq f_i(\Gamma_2)$. 
Graphs $\Gamma \in \text{Graphs}(n)$ are determined by their edges.

**Define:** edge functions $e_{ij} : \text{Graphs}(n) \to \mathbb{F}_2$ for $i < j$.

Polynomial ring: $R = \mathbb{F}_2[e_{12}, \ldots , e_{(n-1)n}]$

$G = S_n$ acts on $\text{Graphs}(n)$ by permuting the vertex labels.

Induced action on the edge functions: $\sigma(e_{ij}) = e_{\sigma(i)\sigma(j)}$.

Invariant functions, $R^G \subseteq R$, are finitely generated (Hilbert's Basis Theorem) and separate isomorphism classes.
Example 1 There are 8 labeled graphs on 3 vertices. However, there are only four isomorphism types of such graphs (in this simple case, they are determined by the number of edges in the graph). The invariant functions

\[ R^G = \mathbb{F}_2[e_{12} + e_{13} + e_{23}, e_{12}e_{13} + e_{12}e_{23} + e_{13}e_{23}, e_{12}e_{13}e_{23}] \]

separate Graphs(3).
Here our polynomial ring was defined over a finite field $\mathbb{F}_2$.

19th century: Invariant Theory developed for fields of characteristic zero.

In prime characteristic we have the Frobenius map $r \rightarrow r^p$.

Frobenius map is closely linked to the Steenrod Algebra.

The Steenrod Algebra

Notation: $R = k[x_1, \ldots, x_n]$, $k = \mathbb{F}_q$, $q = p^e > 0$.

Define: $\phi : R \rightarrow R[[t]]$ on each $x_i$ via

$$x_i \mapsto x_i + x_i^q t$$

and extend $\phi$ to a ring homomorphism. For example,

$$x_i x_j \mapsto (x_i x_j) + (x_i^q x_j + x_i x_j^q)t + (x_i^q x_j^q)t^2.$$  

Define operators $Q_i : R \rightarrow R$ to be the $k$-linear maps defined by applying $\phi$ and then extracting the coefficient of $t_i$ from the resulting polynomial. For instance,

$$Q_1(x_i x_j) = x_i^q x_j + x_i x_j^q.$$
The operators $Q_i$ satisfy a product rule that is an extension of the rule we teach in Calculus:

**Theorem 2 (Cartan’s Formula)** $Q_k(fg) = \sum_{i+j=k} Q_i(f)Q_j(g)$.

**Ex.** $Q_1(x_ix_j) = Q_1(x_i)Q_0(x_j) + Q_0(x_i)Q_1(x_j) = x_i^qx_j + x_ix_j^q$.

$Gl(n, \mathbb{F}_q)$ acts on $R$ by linear change of variables. This action commutes with the $Q_i$.

[Reason: action of $g$ on $\{x_1, \ldots, x_n\}$ represented by same matrix as action on $\{x_1^q, \ldots, x_n^q\}$.]
Theorem 3  The operators $Q_i$ commute with $Gl(n, \mathbb{F}_q)$ action.

If $G$ is any group that acts linearly on $R$, then the Steenrod operations $Q_i$ induce maps $R^G \rightarrow R^G$: if $r \in R^G$ and $g \in G$, then $g \cdot Q_i(r) = Q_i(g \cdot r) = Q_i(r)$.

The $Q_i$ raise degree and preserve invariants. So they can be used to create new (higher degree) invariants from known invariants.
Research on the Steenrod Algebra

• Generators and Relations (the Adem relations).

• Structure of $R^G$ as a module over the Steenrod Algebra.

• Interpretation of the Steenrod Algebra in terms of differential operators.
Algebraic Geometry

Algebraic geometers think of rings in a geometric way (and sometimes use algebra to formalize notions in geometry).

To explain this, consider polynomial functions on $\mathbb{C}^n$.

These can be added and multiplied, so they form a ring, $R = \mathbb{C}[x_1, \ldots, x_n]$. 
**Def:** A variety is the common zero set of a collection of polynomial equations, $f_1 = 0, \ldots, f_k = 0$.

- The union of the two axes in $\mathbb{C}^2$ is just the zero set of $x_1x_2$.
- The variety $X = \mathbb{V}(x_2 - x_1^2, x_2 - 2)$ is just the intersection of a parabola with a line – two points.

What are the functions on a variety $V$?

Polynomials in $(f_1, \ldots, f_k)$ restrict to zero on $V$ so the ring of functions on $V$ is

$$\mathbb{C}[V] = \frac{\mathbb{C}[x_1, \ldots, x_n]}{(f_1, \ldots, f_k)}.$$
This association is really an antiequivalence of categories: polynomial maps of varieties

$$\pi : X \to Y$$

correspond to ring homomorphisms

$$\mathbb{C}[Y] \to \mathbb{C}[X]$$

where \( f \in \mathbb{C}[Y] \) is sent to \( f \circ \pi \).

Rings also give rise to geometric objects: \( R \mapsto \text{Spec}(R) \).

Maximal ideals in \( \mathbb{C}[V] \) correspond to points on \( V \). [But \( \text{Spec}(R) \) contains more points too!]

Algebraic Geometry
The Ring $\mathbb{C}[[t]]$

Question: What is the geometric interpretation of the ring $\mathbb{C}[[t]]$?

There is only one maximal ideal $(t)$ in this ring: $\text{Spec}(\mathbb{C}[[t]]) = pt$.

The ring consists of formal power series centered at zero. Convergence in a microlocal neighbourhood of zero.

We think of $\text{Spec}(\mathbb{C}[[t]])$ as a point, together with a infinitesimal tangent vector, an infinitesimal second-order tangent vector, etc.

The image of a map $\text{Spec}(\mathbb{C}[[t]]) \to X$ is then just a formal arc in $X$: a point in $X$ plus a microlocal curve through the point.
Higher Derivations

Let $R = k[x_1, \ldots, x_n]/I$ be a finitely generated ring.

**Definition:** A higher derivation from $R$ to itself is an infinite collection of maps of $k$-algebras $\{D_0 = id_R, D_1, D_2, \ldots\}$ from $R$ to $R$ that patch together using the product rule:

$$D_k(fg) = \sum_{i+j=k} D_i(f)D_j(g).$$
Examples

Example 4 The Steenrod operators \( \{Q_0, Q_1, \ldots \} \) determine a higher derivation from \( R = k[x_1, \ldots, x_n] \) to itself.

Example 5 In characteristic zero, any derivation \( d \) on \( R \) determines a higher derivation

\[
D_k = \frac{1}{k!} d^k.
\]

For instance, the derivation \( \frac{d}{dx} \) on \( k[x] \) induces a higher derivation on the polynomial ring.
Higher derivations are supposed to extend our notion of derivations (maps satisfying the usual product rule $D(fg) = fD(g) + D(f)g$).

The derivations are represented by the module $\Omega_{R/k}$ of $k$-differentials on $R$.

Each derivation $D : R \to R$ is determined by $\phi : \Omega_{R/k} \to R$ such that $\phi \circ d = D$. 
There is a similar construction that produces a $k$-algebra $HS_{R/k}$ representing the higher derivations.

There is a sequence of maps $(d_0, d_1, \ldots)$ from $R$ to $HS_{R/k}$ such that each higher derivation $\{D_0, D_1, \ldots\}$ from $R$ to $R$ is determined by a unique map $\phi : HS_{R/k} \to R$ of $k$-algebras via $\phi \circ d_i = D_i$. 
**Higher Derivations to** $k$

We can also consider higher derivations from $R$ to $k$.

The whole theory goes through as before: higher derivations are collections of $k$-algebra maps

$$D_i : R \to k$$

with

$$D_k(fg) = \sum_{i+j=k} D_i(f)D_j(g).$$

These are once again determined uniquely by a $k$-algebra map $\phi : HS_{R/k} \to k$.

So we see that the collection of higher derivations $Der_k(R, k)$ is isomorphic to $Hom_{k-alg}(HS_{R/k}, k)$. 
On the other hand, each higher derivation \( \{D_0, D_1, \ldots \} \) from \( R \) to \( k \) determines a map of \( k \)-algebras \( \phi : R \to k[[t]] \) given by:

\[
\phi(r) = D_0(r) + D_1(r)t + D_2(r)t^2 + \cdots
\]

This ring homomorphism is determined by the images \( \phi(x_i) \) of the coordinate functions \( x_i \).

We require that \( f(\phi(x_1), \ldots, \phi(x_n)) = 0 \) for all polynomials \( f \) in the defining ideal \( I \).

So \( \text{Der}_k(R, k) \) is isomorphic to \( \text{Hom}_{k-\text{alg}}(R, k[[t]]) \).
$\text{Der}_k(R, k)$ is isomorphic to $\text{Hom}_{k-\text{alg}}(\text{HS}_{R/k}, k)$.

$\text{Der}_k(R, k)$ is isomorphic to $\text{Hom}_{k-\text{alg}}(R, k[[t]])$.

Thus: $\text{Hom}_{k-\text{alg}}(R, k[[t]]) \cong \text{Hom}_{k-\text{alg}}(\text{HS}_{R/k}, k)$.

Taking Spec’s:

$[\text{Spec}(k[[t]]) \to \text{Spec}(R)] \cong [\text{Spec}(k) \to \text{Spec}\text{HS}_{R/k}] \cong \text{Spec}\text{HS}_{R/k}$.

Jet space $J(\text{Spec}(R)) := \text{Spec}(\text{HS}_{R/k})$ parameterizes arcs on $\text{Spec}(R)$. 
The Jet Space over $X$

Natural map $\pi : J(Spec(R)) \to Spec(R)$ sends each arc to the closed point it passes through.

Each arc $\gamma$ gives rise to a map $R \to k[[t]]$.

This assigns a power series $a_i + HOT(t)$ to each coordinate function $x_i$.

Just looking at the constant terms of these power series gives a point $\pi(\gamma) = (a_1, \ldots, a_n)$ in $k^n$ that lies on $X = Spec(R)$.

$J(X)_O :=$ the set of arcs through $O = \pi^{-1}(O)$. 

Applications of Jet Spaces

- Nash’s Conjecture
- Motivic Integration
Nash’s Conjecture

Jet Space of $X \leftrightarrow$ resolution of singularities of $X$.

$X$: surface with an isolated singular point $O$.

$\bar{X}$: minimal resolution of singularities of $X$.

$\gamma: \bar{X} \to X$ gives rise to essential exceptional divisors

$$\pi^{-1}(O) = \cup E_i.$$ 

Nash: components of $J(X)_O \leftrightarrow \{E_i\}$.

Nash conjectured that this is a bijection.

FALSE: Kollar and Ishii.
Motivic Integration

Batyrev’s Conjecture: two birationally equivalent Calabi-Yau manifolds have the same Hodge numbers.

Hodge numbers: \((h_{p,q} = \dim H^p(\Omega^q, X))\) are numerical invariants.

Map: \(X \mapsto \sum_{i,j} h_{p,q} u^p v^q \in \mathbb{Z}[u,v]\)

Kontsevich: theory of motivic integration.

Map above factors through another map (motivic integration): \(\{\text{varieties}\} \rightarrow \mathcal{M}\).

Plan: Show birational C-Y map to same elt of \(\mathcal{M}\).
Theory of Integration

• Space on which to integrate \((J(X))\).

• Integrable sets (cylinder sets).

• Value set for integration: motivic ring \(\mathcal{M}\).

• Change of variables formula: key to Batyrev’s conjecture.
Other Applications of Motivic Integration

- zeta functions

- $p$-adic integration

- string theory

- mirror symmetry

- characterizations of singularities (Berkeley seminar)