Higher Derivations and Invariant Theory

William N. Traves
U.S. Naval Academy

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Thank You.
Outline

1) Invariant Theory and the Steenrod Algebra
2) Rings of Differential Operators
3) Higher Derivations
4) Jet Spaces and Applications
Invariant Theory

- Invariant theory of 19th and 20th centuries focused on characteristic zero and nonmodular cases.

- Characteristic p>0 largely an afterthought – as in commutative algebra more generally.

- But prime characteristic methods are increasingly important.
  - Applications of commutative algebra to combinatorics.
  - New tool: Frobenius map.
    - Algebraic theory of tight closure
      - mimics and extends results from analysis
  - Invariant theory: Steenrod Algebra.
The Steenrod Algebra

My thanks to Reg Wood for several nice lectures on the Steenrod algebra.

Like Reg, I will consider the Steenrod algebra from an algebraic point of view (as in Larry Smith’s book).

Fix some notation:

- $k = \text{GF}(q) = \mathbb{F}_q$
- $q = p^s$
- $R = k[x_1, \ldots, x_n]$
- $G$: subgroup of $\text{GL}(n,k)$ acting linearly on $R$. 
Steenrod Algebra from $\phi$

$R = k[x_1, \ldots, x_n]$  

$(= k[x,y] \text{ or } k[x,y,z])$

$\varphi: R \rightarrow R[[t]]$

$x_i \mapsto x_i + x_i^q t$

$\varphi(xy) = (x + x^q t)(y + y^q t) = xy + (xy^q + x^q y) t + x^q y^q t^2$

$Q_i : R \rightarrow R$ are the $i^{th}$ Steenrod operators obtained by applying $\varphi$ and extracting the coefficient of $t^i$

$Q_1(xy) = xy^q + x^q y$

$A := \text{Steenrod Algebra} – \text{the } k\text{-algebra generated by the } Q_i.$
Properties of the Operators

**Cartan Formula:** \[ Q_k(fg) = \sum_{i+j=k} Q_i(f)Q_j(g) \]

**Instability:** \[ Q_k(f_d) = \begin{cases} f^q & \text{if } k = d \\ 0 & \text{if } k > d \end{cases} \]

Example: \[ Q_1(xy) = Q_0(x)Q_1(y) + Q_1(x)Q_0(y) \]
\[ = xy^q + x^qy \]

None of the operators are zero, though they are all nilpotent.
The $Q_i$ Commute with $G$

The Steenrod operators $Q_i$ commute with the group action (linear change of variables).

Check this directly. The key fact is that the matrix representing the action of $g \in G$ on $\{x_1, \ldots, x_n\}$ is the same matrix that represents the action of $g$ on $\{x_1^q, \ldots, x_n^q\}$ (because $a^q = a$ in $\mathbb{F}_q$).

The $Q_i$ raise degree and preserve invariants so they create new (higher degree) invariants from old.
Q_i and Frobenius

The Q_i also satisfy an interesting relation with regard to the Frobenius map:

\[ Q_i(r^{p^e}) = \begin{cases} 
(Q_{i/p^e}(r))^{p^e} & \text{if } i, \\
0 & \text{otherwise.}
\end{cases} \]

\[ r^{p^e} + Q_1(r^{p^e})t + Q_2(r^{p^e})t^2 + \cdots = \varphi(r^{p^e}) \]
\[ = (\varphi(r))^{p^e} \]
\[ = (r + Q_1(r)t + Q_2(r)t^2 + \cdots)^{p^e} \]
\[ = r^{p^e} + Q_1(r)^{p^e} t^{p^e} + Q_2(r)^{p^e} t^{2p^e} + \cdots \]

Equating coefficients of t gives the result.
**\( R^{p^e} \)-linearity of the \( Q_i \)**

The \( Q_i \) also satisfy an interesting relation with regard to the Frobenius map:

\[
Q_i (r^{p^e}) = \begin{cases} 
(Q_{i/p^e}(r))^{p^e} & p^e \mid i, \\
0 & \text{otherwise}.
\end{cases}
\]

Now for \( p^e > i \) we have:

\[
Q_i (r^{p^e}f) = \sum_{m+n=i} Q_m (r^{p^e}) Q_n (f) \\
= Q_0 (r^{p^e}) Q_i (f) \\
= r^{p^e} Q_i (f)
\]
Complete set of invariants – the Adem relations – is known.

These are encoded by the Bullett-Macdonald identity.

Much is also known about the structure of $R^G$ as a module over the Steenrod Algebra

- Invariant ideals in $R^G$ – e.g. radical of a stable ideal is stable
- For example, when $G = \text{GL}(n,k)$, there are only finitely many stable prime ideals in $R^G$ and these are generated by intervals in the Dixon invariants

The Steenrod algebra can also be interpreted as a subring of the ring of differential operators on $R^G$. 
Rings of Differential Operators

**Grothendieck**: defined differential operators in an abstract way – subring of \( \text{End}(R) \) satisfying certain iterated commutator relations.

Case \( R = \mathbb{C}[x_1, \ldots, x_n] \):

\[
D(R) = \mathbb{C}[x_1, \ldots, x_n, d_1, \ldots, d_n]
\]

generators satisfy the product rule:

\[
[d_i, x_j] = d_i x_j - x_j d_i = \delta_{ij}
\]

In this case, \( D(R) \) is the **Weyl Algebra** (see Coutinho’s nice book).
Problems in Characteristic $p>0$

Julia Hartmann mentioned some of the problems with differential operators in prime characteristic:

$$d_1(x^p) = p x^{p-1} = 0$$

Even worse: $d_1^p = 0$

Introduce the divided powers operators:

$$d_i^k = \frac{1}{k!} \frac{\partial^k}{\partial x_i^k}$$

Then set $D(k[x_1, ..., x_n]) = k[x_i, d_i^m]_{m>0}$
Differential Operators on $R^G$

We are in the case where $G$ is reductive so think of $R^G$ as $S/I$.

\[
D(R^G) = D(S/I) = \left\{ \theta \in D(S) : \theta(I) \subseteq I \right\}/I \cdot D(S).
\]

**Remark 1**: With this definition, it is not clear that $D(R^G)$ enjoys any nice properties.

**Remark 2**: We could also define $D(R_G)$ in this way.
Theorem (K.E. Smith): In characteristic $p > 0$, the ring of differential operators on a ring $R$ is just the algebra of maps $R \to R$ that are $R^p$-linear for some power $p^e$.

Cor: The Steenrod algebra is a subalgebra of $D(R^G)$.

In fact, we can write the Steenrod operators as

$$Q_i = \sum_{|a|=i} x^{q_a} d^a$$

$$= \sum_{a_1 + \cdots + a_n = i} x_1^{q_{a_1}} \cdots x_n^{q_{a_n}} \frac{1}{a_1! \cdots a_n!} \frac{\partial^i}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}}$$
Applications

Rings of differential operators find application in a wide variety of mathematical fields:

- Model quantum mechanics
- Used to study symplectic manifolds
- Local cohomology modules are finite over $D(R)$
- Close – but mysterious – connections to tight closure
The group $G$ acts on $R$ and this action extends to operators:

$$g \in G, \ d \in D(R) \implies (gd)(r) = gd(g^{-1}r)$$

Note that if $gd = d$ then $d$ defines an operator on $R^G$:

$$g \in G, \ r \in R^G \implies gd(r) = gd(g^{-1}r) = (gd)(r) = d(r)$$

**Natural map:** $D(R)^G \to D(R^G)$
Questions about $D(R^G)$

- When is $D(R^G)$ finitely generated, or left or right Noetherian?
- When is $D(R^G)$ a simple ring?
- When is $R^G$ a simple module over $D(R^G)$?
- What about the same questions for $GrD(R^G)$?
- When is the map $D(R)^G \rightarrow D(R^G)$ surjective?
Answers: $G$ finite

Characteristic zero.

Kantor and Levasseur: $D(R^G)$ is

- finitely generated,

- left and right Noetherian.

- moreover: $D(R)^G \to D(R^G)$ is a surjection whenever $G$ contains no pseudoreflections.
Answers: Classical Groups

Schwarz, Levasseur, Van den Bergh, Musson, Stafford and many others have studied the classical groups acting on a polynomial ring in characteristic zero.

In most cases $D(R^G)$ – and even $GrD(R^G)$ – are finitely generated and left and right Noetherian. In all these cases, the map $D(R)^G \rightarrow D(R^G)$ is surjective. For example, this holds for tori, $O(n)$, etc.

However, Schwarz has shown that there are representations of $Sl_2(C)$ for which the map $D(R)^G \rightarrow D(R^G)$ is not surjective.
Simplicity

If $D(S)$ is simple then the ring $S$ is simple as a $D(S)$-module.

Proof:
- If $I$ is a nonzero stable ideal in $S$ then $S/I$ is a $D(S)$-module.
- $\text{Ann}_{D(S)}(S/I)$ is a two-sided ideal in $D(S)$ that contains $I$.
- So $\text{Ann}_{D(S)}(S/I) = D(S)$.
- Thus $S/I = 1(S/I) = 0$ and $I = S$.
- So $S$ contains no proper $D(S)$-modules.

General feeling: characteristic $p$ is harder than char. 0

Theorem (K.E. Smith and Van den Bergh): In prime characteristic $D(R^G)$ is always a simple ring.
Higher Derivations

The Steenrod operations are an example of a higher derivation, collections of operators that generalize the behavior of derivations on commutative rings.

Let \( R = k[x_1, \ldots, x_n]/I \)

**Definition**: A higher derivation from \( R \) to \( R \) is an infinite collection of \( k \)-algebra maps \( \{D_0 = \text{id}_R, D_1, D_2, \ldots \} \) from \( R \) to \( R \) that patch together using the product rule

\[
D_k (fg) = \sum_{i+j=k} D_i(f)D_j(g)
\]
Examples of Higher Derivations

(1) The Steenrod operators \( \{Q_0, Q_1, \ldots \} \) determine a higher derivation from \( R=k[x_1, \ldots, x_n] \) to itself.

(2) In characteristic zero, any derivation \( d \) on \( R \) determines a higher derivation

\[
D_k = \frac{1}{k!} d^k
\]

For instance, the derivation \( d/dx \) on \( k[x] \) induces a higher derivation on the polynomial ring.
Exponential Maps

Each higher derivation \( \{D_0, D_1, D_2, \ldots \} \) from \( R \) to \( R \) gives rise to a map of \( k \)-algebras

\[
\varphi: R \to R[[t]] \\
r \mapsto D_0(r) + D_1(r)t + D_2(r)t^2 + \ldots
\]

The product rule guarantees that this map is a ring map.

There is no instability result for higher derivations, but each \( D_i \) is \( R^{pe} \)-linear for some power \( pe \). So each higher derivation is a differential operator.
The Higher Derivation Algebra

The higher derivation algebra $HDer(R)$ on a ring $R$ is just the $R$-algebra generated by the components of all higher derivations on $R$.

Larry Smith asked whether $A = HDer$. 

$$R^G \otimes A \subseteq HDer(R^G) \subseteq D(R^G)$$
Case: $R^G$ a Polynomial Algebra

Here $\text{HDer}(R^G) = D(R^G)$ but $R^G A \neq \text{Hder}(R^G)$.

Equality follows from direct calculation. In fact $\text{HDer}(S) = D(S)$ whenever $S$ is smooth over $k$.

Inequality now follows because $R^G$ has $A$-stable ideals (for example, the augmentation ideal), but is $D(R^G)$-simple.
Nakai’s Conjecture

Conjecture: S is smooth over k if and only if $\text{HDer}(S) = D(S)$.

Ishibashi proved Nakai’s conjecture for $R^G$, $G$ a finite group.

When $R^G$ is singular, there is a nice theory of $\text{HDer}(R^G)$-stable ideals, similar to that developed by Smith for $A$-stable ideals. But it remains open whether $\text{HDer}(R^G) = R^G A$ in the singular case.
Derivations are Representable

\[ \text{Higher derivations } (\mathcal{S} \to \mathfrak{K}) \cong \text{Hom}_{k\text{-alg}}(\text{HS}_{S/k}, \mathfrak{K}) \]

\[ \{D_0, D_1, \ldots\} \mapsto \varphi : \forall i \quad D_i = \varphi \circ d_i \]
**Aecs**

Suppose $S = k[x_1, \ldots, x_n]/I$ and $D = \{D_0, D_1, \ldots\}$ is a higher derivation from $S$ to $k$.

Then we get a ring map $\varphi: S \to k[[t]]$ given by

$$\varphi(s) = D_0(s) + D_1(s)t + D_2(s)t^2 + \ldots.$$ 

This map is determined by the images $\varphi(x_i)$.
These need to satisfy $f(\varphi(x_1), \ldots, \varphi(x_n)) = 0$ for each $f$ in the defining ideal $I$.

Higher derivations $(S \to k) \cong \text{Hom}_{k-\text{alg}}(S, k[[t]])$
An Adjointness Result

Theorem: \( \text{Hom}_{k\text{-alg}}(HS_{S/k}, k) \cong \text{Hom}_{k\text{-alg}}(S, k[[t]]) \)

Taking Spec’s:

\[
[\text{maps Spec}(k[[t]]) \to \text{Spec}(S)] \cong [\text{maps Spec}(k) \to \text{Spec}(HS_{S/k})] \\
\cong \text{Spec}(HS_{S/k})
\]

So Spec(HS_{S/k}) parameterizes arcs on Spec(S).
The Jet Space

The Jet Space $J(S) = \text{Spec}(HS_{S/k})$ parameterizes arcs on $\text{Spec}(S)$.

We have a map $pr : J(S) \to \text{Spec}(S)$ that sends each arc to the point it passes through.

Each arc $\gamma$ corresponds to a map $\varphi : S = R/I \to k[[t]]$.

The point $(\varphi(x_1) \mod (t), ..., \varphi(x_n) \mod (t))$ satisfies each equation in $I$, so it lies on $\text{Spec}(S)$. This is the image of the arc $\gamma$ under the map $pr$. 
Applications of Jet Spaces

- Characterization of singularities
  - Multiplier Ideals
  - Nash Conjecture
- Motivic Integration
Nash’s Conjecture

Nash conjectured a relation between the jet space of an algebraic variety $X$ and its resolution of singularities.
Minimal Resolution

$X$: surface with isolated singularity at the origin

$Y$: minimal model for $X$ (blowup and normalize)
The Nash Map (1)

Each arc centered over 0 gives a map Spec(k[[t]]) \rightarrow X. The closed point goes to 0, but the generic point lifts to Y. The VCP ensures that we can complete this map to a map of schemes.

In fact, each arc gets sent into a unique exceptional divisor. 

\( pr^{-1}(0) \subseteq J(X) \rightarrow X \)
In fact, each component of the fiber of arcs through 0 gets sent to a unique exceptional divisor, giving rise to an injective map of sets
\{components\ of\ arc\ space\ through\ 0\} \rightarrow \{exceptional\ divisors\ appearing\ in\ minimal\ resolution\ of\ singularities\}

This map is known as the Nash map. Nash conjectured that this map is a bijection.
Recent Work

The Nash conjecture has motivated research in resolution of singularities (esp. in prime characteristic) for some time [Spivakovsky, Lejeune-Jalabert].

The conjecture is true for toric varieties, surfaces and threefolds.

However, Kollar and Ishii recently gave a counterexample.
Motivic Integration

Kontsevich developed motivic integration to prove a conjecture of Batyrev: *Two birationally equivalent Calabi-Yau manifolds have the same Hodge numbers* \((h_{p,q} = \dim H^p(\Omega^q, X))\).

Get a map that sends \(X\) to \(\sum h_{p,q} u^p v^q \in \mathbb{Z}[u,v]\).

Kontsevich shows this map factors through another map \(X \to M\) \((M\) is the motivic ring \([BP]\)).
The Motivic Ring

The motivic ring $M$ consists of $\mathbb{Z}$-linear combinations of varieties plus some formal inverses. Sums correspond to disjoint unions and products correspond to direct products.

Kontsevich views the map $X \to M$ as an integration on the arc space of $X$. He gets a change of variables formula that he uses to show that the integrals of $X$ and $Y$ (birational CY manifolds) are equal. Then their Hodge numbers are equal too.
Applications: Motivic Integration

- zeta functions
- p-adic integration
- string theory
- mirror symmetry
- multiplier ideals (tight closure): singularity theory
  - Ein, Lazarsfeld and Mustata
Thank you once again.

Enjoy your lunch!