Higher Derivations in Invariant Theory

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Much of the wonderful invariant theory of the 19th century worked over fields of characteristic zero, while the theory for prime characteristic lagged behind. However, the Frobenius ($p$th power) map in characteristic $p > 0$ leads to a rich theory of invariants in prime characteristic. This theory is closely bound up with the Steenrod Algebra, which allows us to derive new invariants from known invariants.

I won't go into the history of the Steenrod Algebra except to say that it was developed to study certain cohomology rings in Algebraic Topology. We'll take a more elementary (and algebraic) approach to the Steenrod Algebra which has been popularized by L. Smith and R. Wood.

1 The Steenrod Algebra

Notation: Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field $k = \mathbb{F}_q$ of characteristic $p > 0$ (hence $q = p^e$). We think of $R$ as the ring of polynomial functions on the affine space $k^n$.

Define a map $\phi : R \to R[[t]]$ on each $x_i$ via

$$x_i \mapsto x_i + x_i^q t$$

and extend $\phi$ to a ring homomorphism. For example,

$$x_i x_j \mapsto (x_i x_j) + (x_i^q x_j + x_i x_j^q) t + (x_i^q x_j^q) t^2.$$  

Define operators $Q_i : R \to R$ to be the $k$-linear maps defined by applying $\phi$ and then extracting the coefficient of $t_i$ from the resulting polynomial. For instance,

$$Q_1(x_i x_j) = x_i^q x_j + x_i x_j^q.$$
Definition: The operators $Q_i$ form a $k$-subalgebra of the $k$-linear endomorphisms of $R$ called the Steenrod Algebra and denoted $\mathcal{A}$.

The operators $Q_i$ satisfy a product rule that is an extension of the rule we teach in Calculus:

**Theorem 1.1** (Cartan Formula). $Q_k(fg) = \sum_{i+j=k} Q_i(f)Q_j(g)$.

**Example 1.2.** $Q_1(x_i x_j) = Q_1(x_i)Q_0(x_j) + Q_0(x_i)Q_1(x_j) = x_i^q x_j + x_i x_j^q$.

As well, the operators $\{Q_i : i > 0\}$ are unstable in the sense that if $f_d$ is a homogeneous polynomial of degree $d$ then $Q_i(f_d)$ is $f_d^q$ if $i = d$ and is 0 if $i > d$. In particular, none of the operators are zero (though they are all nilpotent!). [check]

Now $Gl(n, \mathbb{F}_q)$ acts on $R$ by linear change of variables. Since the coefficients of the matrices are in $\mathbb{F}_q$, they satisfy $a^q = a$. This implies that the action of $Gl(n, \mathbb{F}_q)$ on the vector space with basis $\{x_1, \ldots, x_n\}$ is the same as its action on the vector space with basis $\{x_1^q, \ldots, x_n^q\}$ (the group action is represented by the same matrix). As a consequence, the Steenrod operations commute with linear group actions. If $G$ is any group that acts linearly on $R$, then the Steenrod operations $Q_i$ induce maps $R^G \rightarrow R^G$: if $r \in R^G$ and $g \in G$, then $g \cdot Q_i(r) = Q_i(g \cdot r) = Q_i(r)$. Here $R^G$ is the ring of invariants of the group action and it should be thought of as the ring of functions on the quotient space $k^n/G$.

The $Q_i$ raise degree and preserve invariants. So they can be used to create new (higher degree) invariants from known invariants. These operators have another important property: for each operator $Q_i$, there is a power $p^e$ so that $Q_i$ is $R^{p^e}$-linear. This follows from Cartan’s Formula and the unstability property: for $p^e > i$ we have

$$Q_i(r^{p^e} f) = \sum_{m+n=i} Q_m(r^{p^e})Q_n(f) = Q_0(r^{p^e})Q_i(f) = r^{p^e}Q_i(f)$$

Much is known about the structure of the Steenrod Algebra. The complete set of relations, called the Adem relations, has been determined. These can be derived in several beautiful ways and are encoded by the Bullet-Macdonald identity. Much is also known about the structure of $R^G$ as a module over $\mathcal{A}$; for instance, there are only finitely many stable prime ideals in $R^G$ and these are generated by intervals in the Dixon invariants.

The Steenrod Algebra can also be interpreted as a ring of differential operators. We take up this theme in the next section.
2 Differential Operators

Grothendieck defined a ring of differential operators on algebraic variety. His definition is abstract and quite complicated, but it can be easily understood for \( k^n \). The ring of differential operators on \( \mathbb{C}^n \) (or equivalently, on \( R = \mathbb{C}[x_1, \ldots, x_n] \)) is \( D(R) = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n] \). This is sometimes called the Weyl algebra. It is not quite a polynomial ring: the generators satisfy the relations

\[
[\partial_i, x_j] = \partial_i x_j - x_j \partial_i = \delta_{ij}.
\]

(This is just the ordinary product rule: \( \partial_i = \frac{\partial}{\partial x_i} \).)

When \( k \) has prime characteristic \( p > 0 \), something odd happens:

\[
\partial_i(x_i^p) = px_i^{p-1} = 0.
\]

Moreover, \( \partial_i^p \equiv 0 \). To get around this problem, we introduce the divided powers operators, \( \partial_i^m = \frac{1}{m!} \frac{\partial^m}{\partial x_i^m} \). We have \( \partial_i^m(x_i^r) = \frac{i(i-1)\cdots(i-m+1)}{m!} x_i^{r-m} = \binom{i}{m} x_i^{r-m} \). Then

\[
D(k[x_1, \ldots, x_n] = k[x_i, \partial_i^m]_{m>0}.
\]

We will want to discuss the relation between \( D(R) \) and \( D(R^G) \) so we need to define the ring of differential operators on affine varieties. There are two equivalent ways to do this. If we think of \( R^G \) as \( S/I \) then

\[
D(R^G) = D(S/I) = \{ \theta \in D(S) : \theta(I) \subseteq I \} / ID(S).
\]

Alternatively, we could think of \( R^G \) as a subalgebra of \( R \). From this point of view

\[
D(R^G) = \{ \theta \in D(R) : \theta(R^G) \subseteq R^G \} / \{ \theta \in D(R) : \theta(R^G) = 0 \}.
\]

There is a third description of the ring of differential operators that makes it clear that the \( Q_i \) are differential operators.

**Theorem 2.1** (K. E. Smith). In characteristic \( p > 0 \), the ring of differential operators on \( R \) is just the subalgebra of all endomorphisms of \( R \) that are \( R^p \)-linear for some power \( p^e \).
Since $\mathcal{A}$ is generated by $R^p$-linear maps, the Steenrod Algebra is a subalgebra of the ring of differential operators. In fact, we can write

$$Q_i = \sum_a x^{qa} \partial^a.$$ 

If we filter the ring of differential operators by the order of the operators appearing, then $Q_i$ is a differential operator of order $i$.

If $G$ is a group that acts on $R$, then it can be made to act on differential operators too. For $g \in G$ and $D \in D(R)$, set $gD$ to be the operator such that $(gD)(r) = g \cdot D(g^{-1} \cdot r)$. Note that if $g \cdot D = D$ then $g$ commutes with $D$ (and hence $D$ defines an operator on $R^G$).

Rings of differential operators find application in a wide range of mathematical fields. For instance, they can be used to model quantum mechanics, they are used to study symplectic manifolds, and they play an increasingly important role in commutative algebra (through a mysterious relationship with tight closure).

Questions: When is the ring of differential operators $D(R^G)$ finitely generated, left or right Noetherian? When is $D(R^G)$ a simple ring? And when is $R^G$ a simple module over $D(R^G)$? What about the same questions for $GrD(R^G)$? How do the algebras $D(R^G)$ and $D(R^G)$ relate?

Kantor and Levasseur studied the case when $G$ is a finite group. In this case $D(R^G)$ is finitely generated, simple and left and right Noetherian. Moreover the map $\pi : D(R^G) \rightarrow D(R^G)$ is a surjection whenever $G$ contains no pseudoreflections.

Schwarz, Levasseur, Van den Bergh, Musson, Stafford and many others have studied the classical groups acting on a polynomial ring in characteristic zero. Here, $D(R^G)$ (and even $GrD(R^G)$) are known to be finitely generated when $G$ is commutative. As well, the map $D(R)^G \rightarrow D(R^G)$ is a surjection. However, there are representations of $Sl_2(\mathbb{C})$ for which the map $D(R)^G \rightarrow D(R^G)$ is not a surjection (Schwarz).

The question of simplicity is quite interesting. If $D(S)$ is simple, then $S$ is a simple module over $D(S)$. [Reason: $Ann_{D(S)}(S/I)$ is a nonzero two-sided ideal of $D(S)$ and hence must be all of $D(S)$ – i.e. $I = S$.]

There is a general feeling (perhaps misguided) that the characteristic zero case is better than the prime characteristic case. However, when $R$ has prime characteristic, $D(R^G)$ is always simple. [due to Van den Bergh and K. E. Smith] This remains an open problem in general in characteristic zero.
3 Higher Derivations

The Steenrod operations are an example of a higher derivation, collections of operators that generalize the behaviour of derivations on commutative rings.

Let \( R = k[x_1, \ldots, x_n]/I \) be a finitely generated ring. A higher derivation from \( R \) to itself is an infinite collection of maps of \( k \)-algebras \( \{D_0 = id_R, D_1, D_2, \ldots\} \) from \( R \) to \( R \) that patch together using the product rule:

\[
D_m(fg) = \sum_{i+j=m} D_i(f)D_j(g).
\]

**Example 3.1.** The Steenrod operators \( \{Q_0, Q_1, \ldots\} \) determine a higher derivation from \( R = k[x_1, \ldots, x_n] \) to itself.

**Example 3.2.** In characteristic zero, any derivation \( d \) on \( R \) determines a higher derivation

\[
D_k = \frac{1}{k!}d^k.
\]

For instance, the derivation \( \frac{d}{dx} \) on \( k[x] \) induces a higher derivation on the polynomial ring.

Each higher derivation from \( R \) to \( R \) gives rise to a map of \( k \)-algebras

\[
\phi : R \to R[[t]]
\]

\[
r \mapsto D_0(r) + D_1(r)t + D_2(r)t^2 + \cdots
\]

analogous to the case of the Steenrod operators.

There is no unstability result for the higher derivations in general, but we do have the following nice fact in prime characteristic \( p > 0 \):

\[
D_m(r^p) = \begin{cases} 
(D_{m/p^e}(r))^{p^e} & \text{if } p^e \mid m \\
0 & \text{otherwise}
\end{cases}
\]

**Proof:** Apply the ring homomorphism \( \phi \) to \( r^p \):

\[
r^p + D_1(r^p)t + D_2(r^p)t^2 + \cdots = \phi(r^p) = (\phi(r))^{p^e} = (r + D_1(r)t + D_2(r)t^2 + \cdots)^{p^e}
\]

\[
= r^{p^e} + D_1(r)^{p^e}t^{p^e} + D_2(r)^{p^e}t^{2p^e} + \cdots
\]

**Theorem 3.3.** Each higher derivation is a collection of differential operators.
Proof: The previous result, combined with the product rule, implies that each $D_i$ is $R_p^e$-linear for some power $p^e$.

The algebra of higher derivations a ring $S$ is just the $S$-algebra $HDer(S)$ generated by the components of all higher derivations on $S$. We have

$$R^G A \subseteq HDer(R^G) \subseteq D(R^G).$$

Here, $RA$ is the $R^G$-algebra generated by the Steenrod operators.

It makes sense to ask when these inclusions become equalities. When $R^G$ is regular (e.g., a polynomial algebra), $R^G A \neq HDer(R^G)$ but $HDer(R^G) = D(R^G)$. The first statement follows from the second since in this case $D(R^G)$ is a simple ring so $R^G$ is a simple module over $HDer(R^G) = D(R^G)$ but $R^G$ contains a nontrivial $RA$-stable ideal generated by the Dixon invariants. The fact that $HDer(R^G) = D(R^G)$ when $R^G$ is regular follows from a delicate argument: first prove the result for polynomial rings and then lift the result to $Spec(R^G)$, which is an étale cover of an affine space.

**Conjecture 3.4** (Nakai’s Conjecture). $HDer(S) = D(S)$ if and only if $S$ is regular.

The conjecture was proven for $S = R^G$ a ring of invariants on a polynomial ring under a finite group action (Ishibashi). This is easy to see in the prime characteristic case where $R^G$ is not regular because $D(R^G)$ is a simple algebra but $HDer(R^G)$ is not: the singular locus of $R$ corresponds to a radical ideal that is $HDer$-stable. In this case, I do not know when $RA$ can equal $HDer(R^G)$.

### 4 Jet Spaces

Higher derivations are supposed to extend our notion of derivations (maps satisfying the usual product rule $d(fg) = fd(g) + d(f)g$). The derivations are represented by the module $\Omega_{R/k}$ of $k$-differentials on $R$. That is, there is a map $d : R \rightarrow \Omega_{R/k}$ such that whenever $D : R \rightarrow R$ is a derivation, there exists a unique $R$-module homomorphism $\phi : \Omega_{R/k} \rightarrow R$ such that $\phi \circ d = D$. We say that $\Omega_{R/k}$ represents the derivations since each derivation is determined by $\phi \in Hom_R(\Omega_{R/k}, R)$.

There is a similar construction that produces a $k$-algebra $HS_{R/k}$ representing the higher derivations. Analogous to the derivation case, there is a
sequence of maps \((d_0, d_1, \ldots)\) from \(R\) to \(HS_{R/k}\) such that each higher derivation \(\{D_0, D_1, \ldots\}\) from \(R\) to \(R\) determines a unique map \(\phi : HS_{R/k} \to R\) of \(k\)-algebras by \(\phi \circ d_i = D_i\).

### 4.1 Higher Derivations to \(k\)

We can also consider higher derivations from \(R\) to \(k\). The whole theory goes through as before: higher derivations are collections of \(k\)-algebra maps \(D_i : R \to k\) with \(D_k(fg) = \sum_{i+j=k} D_i(f)D_j(g)\). These are once again determined uniquely by a \(k\)-algebra map \(\phi : HS_{R/k} \to k\).

So we see that the collection of higher derivations \(HDer_k(R, k)\) is isomorphic to \(Hom_{k-alg}(HS_{R/k}, k)\), the collection of \(k\)-algebra maps from \(HS_{R/k}\) to \(k\). (*)

On the other hand, each higher derivation \(\{D_0, D_1, \ldots\}\) from \(R\) to \(k\) determines a map of \(k\)-algebras \(\phi : R \to k[[t]]\) given by:

\[
\phi(r) = D_0(r) + D_1(r)t + D_2(r)t^2 + \cdots
\]

This ring homomorphism is determined by the images \(\phi(x_i)\) of the variables \(x_i\). We require that \(f(\phi(x_1), \ldots, \phi(x_n)) = 0\) for all polynomials \(f\) in the defining ideal \(I\).

Thus, \(HDer_k(R, k)\) is isomorphic to \(Hom_{k-alg}(R, k[[t]])\). (**) From (*) and (**) we get that \(Hom_{k-alg}(R, k[[t]]) \cong Hom_{k-alg}(HS_{R/k}, k)\).

Taking Spec’s we see that

\[[\text{maps } Spec(k[[t]]) \to Spec(R)] \cong [\text{maps } Spec(k) \to SpecHS_{R/k}] \cong SpecHS_{R/k}\].

Thus, the jet space \(J(\text{Spec}(R)) := Spec(HS_{R/k})\) parameterizes arcs on \(\text{Spec}(R)\). We have a natural map \(\pi : J(\text{Spec}(R)) \to Spec(R)\) that sends an arc to the closed point that it passes through. We can understand this map using the ring map. Each arc \(\gamma\) gives rise to a map \(R \to k[[t]]\) (here we are using the antiequivalence of categories between schemes and rings). This assigns a power series \(a_i + HOT(t)\) to each coordinate function \(x_i\). Just looking at the constant terms of these power series gives a point \(\pi(\gamma) = (a_1, \ldots, a_n)\) in \(k^n\) that lies on \(X = Spec(R)\). We call the preimage of a point \(O\) under this map \(\pi\), the set of arcs through \(O\) (it is actually a scheme).
4.2 Nash’s Conjecture

Nash conjectured a relationship between the jet space and resolution of singularities. To state the conjecture, suppose that $X$ is an algebraic surface with an isolated singular point at the origin $O$. It is a classical result that we can resolve the singularities of $X$ by repeatedly blowing up points (which can introduce 1-dimensional singularities) and then normalizing (which replaces the 1-dimensional singularities by singularities at points). This leads to a smooth surface $\bar{X}$ called the minimal model of $X$. It is minimal in the sense that if any other variety $Z$ maps to $X$ and is birational, then the map must factor through $\gamma: \bar{X} \rightarrow X$.

The essential exceptional divisors of $X$ are just the preimage $\gamma^{-1}(O)$ of the isolated singularity. This is a union of curves $\cup E_i$ on the minimal model $\bar{X}$. Nash produced an injective map from the components of the jet space of arcs on $X$ through $O$ to the set of exceptional divisors. He conjectured that this map is surjective. This conjecture is true for many classes of varieties (eg. toric varieties, surfaces, etc), however Kollar and Ishii recently showed that the conjecture is false in general.

4.3 Motivic Integration

The jet space also plays a crucial role in Kontsevich’s theory of motivic integration. He developed this theory to prove a conjecture of Batyrev: two birationally equivalent Calabi-Yau manifolds have the same Hodge numbers. For the purposes of this overview, it is not really important to know what Calabi-Yau manifolds are, or to know what Hodge numbers are. All we need to know is that the Hodge numbers $(h_{p,q} = \dim H^p(\Omega^q, X))$ are numerical invariants of $X$. We have a map that sends each Calabi-Yau variety $X$ to $\sum_{p,q} h_{p,q} u^p v^q \in \mathbb{Z}[u,v]$.

Kontsevich showed that this map factors through another map $X \rightarrow \mathcal{M}$, that sends the variety $X$ to an element in a ring that he calls the motivic ring. The ring $\mathcal{M}$ consists of formal $\mathbb{Z}$-linear combinations of subvarieties in $X$ enlarged by adding inverses of some of the elements; addition corresponds to disjoint union of varieties while multiplication corresponds to direct product of varieties. Kontsevich interprets this map to the motivic ring as an integration map on the arc space of $X$. Roughly, Kontsevich develops a new theory of integration in which the space we integrate over is the arc space of $X$, the measurable sets are certain subsets of $J(X)$ called cylinder sets, the
integrals take values in this odd motivic ring and we have a nice change of variables formula for integration. He uses this change of variables formula to show that when $X$ and $Y$ are birational Calabi-Yau manifolds their images in the motivic ring are equal. Thus, so are their images in the ring $\mathbb{Z}[u, v]$.

Motivic integration also finds many other interesting applications, including zeta functions, $p$-adic integration, string theory and mirror symmetry. There is a nice seminar at Berkeley this year that is investigating applications of motivic integration to characterizations of singularities.