Geometry: Old and New

Will Traves

Department of Mathematics
United States Naval Academy*

Roanoke College
MAA MD/DC/VA Section Meeting
25 APR 2015

* Any views or opinions presented in this talk are solely those of the presenter
and do not necessarily represent those of the U.S. Government
Ancient Geometry

Line Arrangement due to Pappus of Alexandria (Synagogue; c. 340)

Richter-Gebert: 9 proofs in Perspectives on Projective Geometry, 2011
Pascal’s Mystic Hexagon Theorem

Pascal: placed the 6 intersection points on a conic (1639)

Braikenridge and MacLaurin: proved the converse
Why Mystic?
The Projective Plane

$\mathbb{P}^2$ is a compactification of $\mathbb{R}^2$

$\mathbb{P}^2 = \mathbb{R}^2 \cup \text{ line at } \infty$

Parallel lines meet at infinity - one point at $\infty$ for each slope.
Line at infinity wraps twice around $\mathbb{R}^2$. 
\( \mathbb{P}^2 \) is a compactification of \( \mathbb{R}^2 \)
\[
\mathbb{P}^2 = \mathbb{R}^2 \cup \text{ line at } \infty
\]

Parallel lines meet at infinity - one point at \( \infty \) for each slope.
Line at infinity wraps twice around \( \mathbb{R}^2 \).
Topology of the Projective Plane

Thicken line at infinity:
\( \mathbb{P}^2 = \text{disk} \cup \text{Möbius band} \)

\( \mathbb{P}^2 \) can’t be embedded in \( \mathbb{R}^3 \)
(Conway, Gordon, Sachs (1983): linked triangles in \( K_6 \))
$K_6$ embedded in $\mathbb{P}^2$

No linked triangles
Bézout’s Theorem

Compactness of $\mathbb{P}^2$ allows us to count solutions:

**Theorem (Bézout)**

*Any two curves, without common components, defined by the vanishing of polynomials of degrees $d_1$ and $d_2$ meet in $d_1 d_2$ points in $\mathbb{P}^2$, suitably interpreted.*

\[4x^2 + 9y^2 = 36\]

Lines meeting an Ellipse

\[y = 0\]

\[y = 1\]

\[y = 2\]

\[y = 3\]

Line meets curve in two points (possibly imaginary).

Double point: tangency when $y=2$

\[4x^2 + 9(4) = 36\] so $4x^2 = 0$
Projective Coordinates: Möbius’s model for $\mathbb{P}^2$

Möbius: $\mathbb{P}^2 = \{\text{lines through the origin}\}$

Line through $(x, y, z)$ has coordinates $(x : y : z)$ with

$$(x : y : z) \sim (\lambda x : \lambda y : \lambda z) \text{ for } \lambda \neq 0$$

Lines meeting $z = 1$ are of form $(x : y : 1) \sim (x, y)$

Other lines (in $xy$-plane) form points at infinity $(x : y : 0)$
Lines in \( \mathbb{P}^2 \): 1

Lines in \( \mathbb{P}^2 \) correspond to planes through origin.

Line \( \{ (x : 0 : z) \} \) meets line \( \{ (0 : y : z) \} \) at point \((0:0:1)\).
Parallel lines: \( \{(x : 0 : z)\} \) meets \( \{(x : z : z)\} \) at point (1:0:0).
Homogenization

Can’t talk about the parabola $y = x^2$:

If we scale the coordinates $(x : y : z) = (\lambda x : \lambda y : \lambda z)$ then we’d require

$$\lambda y = \lambda^2 x^2$$

for all $\lambda$ (true only for the origin $(x, y) = (0, 0)$).

Curves in $\mathbb{P}^2$ are defined by the vanishing of homogeneous polynomials:

$$y - x^2 = 0 \iff yz - x^2 = 0$$
Enumerative Geometry of Conics: How many conics pass through \( p \) points and are tangent to \( \ell \) lines and \( c \) conics in general position?
Conics through 5 points

\[ ax^2 + bxy + cy^2 + dx + ey + f = 0 \]
\[ ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0 \]
\[(a : b : c : d : e : f) \in \mathbb{R}^6 / \sim = \mathbb{P}^5 \]

Point conditions force \((a : b : c : d : e : f)\) to lie on a hyperplane.

\((x_0 : y_0 : z_0)\) on \(C \iff ax_0^2 + bx_0y_0 + cy_0^2 + dx_0z_0 + ey_0z_0 + fz_0^2 = 0.\)

Intersecting 5 hyperplanes in \(\mathbb{P}^5\) gives a single point corresponding to the one conic through all 5 points.
Given three points in $\mathbb{P}^2$ we can list them as columns of a 3x3 matrix,

$$\begin{bmatrix}
  a_x & b_x & c_x \\
  a_y & b_y & c_y \\
  a_z & b_z & c_z 
\end{bmatrix}.$$ 

The determinant $[abc]$ of this matrix measures six times the volume of the tetrahedron with edges $a$, $b$ and $c$.

$[abc] = 0 \iff$ points $a$, $b$, $c$ are collinear.

Can recover coordinates from full knowledge of determinants.
Plücker Relations

The determinants of the 3x3 submatrices of a larger matrix satisfy quadratic relations.

For example, given five points in $\mathbb{P}^2$ we form the matrix

$$
\begin{bmatrix}
a_x & b_x & c_x & d_x & e_x \\
a_y & b_y & c_y & d_y & e_y \\
a_z & b_z & c_z & d_z & e_z
\end{bmatrix}.
$$

Cramer’s Theorem implies that

$$
[abc][ade] - [abd][ace] + [abe][acd] = 0.
$$

In particular, if $[abc] = 0$ then $[abe][acd] = [abd][ace]$. 
Six points lie on a conic
\[\iff \ [abc][aef][dbf][dec] = [def][dbc][aec][abf].\]

**Conic:**

\[
\begin{array}{c}
\text{[125][136][246][345]} = +[126][135][245][346] \\
[157][259] = -[125][597] \\
[126][368] = +[136][268] \\
[245][297] = -[247][259] \\
[247][268] = -[246][287] \\
[346][358] = +[345][368] \\
[135][587] = -[157][358] \\
[297][587] = +[287][597] \\
\end{array}
\]

If points 2, 5, and 7 are not collinear, \([257] \neq 0\) so \([987] = 0\).
Conjecture (Kepler)

No packing of spheres covers more than $\pi/3\sqrt{2}$ (approximately 74%) of the filled volume.

History: Harriot and Sir Walter Raleigh (1591) and Kepler (1611) Gauss (regular lattices; 1631) and Fejes Tóth (1953) Tom Hales and Sam Ferguson (1992-2006) FlysPecK (Formal Proof of Kepler; 2014)
Prediction: Computers will become our mathematical assistants, vastly raising the level of our mathematical reasoning (c.f. advanced chess and CAD/origami).
We already have strong computer and robotic assistance:

Mind controlled Deka-arm
Wild Conjecture: In the (perhaps distant) future the division between human and computer will become less and less distinct.
There is at least one degree 3 curve through every set of 9 points.

**Question**

*When do 10 points lie on a plane curve of degree 3?*

Smooth curves of degree 3 are called elliptic curves and play a role in both *elliptic curve cryptography* and in Wiles’s proof of *Fermat’s Last Theorem*.

The 10 points lie on a cubic when the determinant of a $10 \times 10$ matrix is zero (25 million terms).

Reiss (1842) wrote out a 20 term polynomial of degree 10 in the $3 \times 3$ brackets that computes the determinant faster.
We developed a straightedge-and-compass construction that checks whether 10 points lie on a cubic.

Such constructive questions are now back in vogue since they leverage all sorts of ideas in computational geometry.
The Key Idea: Cayley-Bacharach

10 points on a cubic precisely when 6 auxiliary points on a conic.

We construct the 6 points using straightedge and compass and then invoke Pascal’s Theorem.
Meet and Join Algebra: Allows algebraic formulation of straightedge-only constructions

Theorem (after Sturmfels and Whiteley)

There exists a straightedge construction to determine if 10 points lie on a cubic (with about 100 million lines).