Enumerative Geometry of Hyperplane Arrangements

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6 DEC 2013
Warm-up Question

Joint work with Max Wakefield and Tom Paul

Question

*How many generic arrangements of $k$ lines pass through $d$ points in general position in $\mathbb{P}^2$?*

If $d$ is too small there are infinitely many arrangements. If $d$ is too large there are no arrangements. If $d = 2k$ then there is a finite number of arrangements through the $d$ points.

No line can pass through 3 points (by general position assumption); PHP shows that each line goes through 2 points.

Count \(\frac{(2k)^2}{k!} = \frac{(2k-1)!}{k!}\)
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Counting Arrangements

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$$\text{Count} = \binom{2k}{2} \binom{2k-2}{2} \cdots \binom{2}{2} / k! = (2k - 1)!!$$
The Intersection Lattice of $\mathcal{A}$

**Problem**

Given a hyperplane arrangement $\mathcal{A}$ in $\mathbb{P}^n$, count the number of hyperplane arrangements with intersection lattice isomorphic to $\mathcal{L}(\mathcal{A})$ that pass through $d$ points in general position.

The intersection lattice (or poset) of $\mathcal{A}$ encodes how the hyperplanes intersect:
The Moduli Space $\mathcal{M}_A$

**Definition**

If $\mathcal{A}$ is an arrangement of $k$ hyperplanes in $\mathbb{P}^n$ then set

$$\mathcal{M}_A := \{\text{arrangements with intersection lattice } \cong \mathcal{L}(\mathcal{A})\} \subset (\mathbb{P}^{n*})^k / S_k$$

with dimension $d = d_A$.

Hyperplane $H : a_0x_0 + \cdots + a_nx_n = 0$ corr. to $(a_0 : \ldots : a_n) \in \mathbb{P}^{n*}$

$\mathcal{A}$ passes through $P \iff (H_1 \cdots H_k)(P) = 0$ – a codim 1 condition

dimension $d = \# \text{ point conditions giving a finite counting problem}$
The degree $N_A$ of $M_A$ is the **number of arrangements** in $M_A$ through $d$ points in general position.

The map

\[
\begin{align*}
(\mathbb{P}^n)^k & \longrightarrow \mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_k) \\
(H_1, \ldots, H_k) & \longmapsto H_1 \cdots H_k
\end{align*}
\]

factors through $(\mathbb{P}^n)^k / S_k$.

The condition that $A$ passes through a point, $(H_1 \cdots H_k)(P) = 0$, is a **linear** condition on $\mathbb{P}(\mathbb{C}[x_0, \ldots, x_n]_k)$ and pulls back to a **multilinear** condition on $(\mathbb{P}^n)^k$. 
Finding $d$ the Easy Way...

Dim $M_A = \#$ degrees of freedom you have in drawing $A$ and represent each degree by a point constraint:

Virtual dimension:

$$vdim(M_A) = 2k - \sum (n_p - 2)$$

with $n_p = \mid \{ H \in A : p \in H \} \mid$
Pappus: \( \text{vdim} = 2(9) - 9 = 9 < 10 = \text{dim} \)
An (Intractible?) Open Problem

Problem

*Find a combinatorial algorithm to compute \( \dim M_A \) from \( \mathcal{L}(A) \).*

Mñev’s Universality Theorem \( \Rightarrow M_A \) arbitrarily complicated

Need to “see” syzygies among defining equations for \( M_A \) from \( \mathcal{L}(A) \).
Cohomological Approach

Goal: count arrangements without precise location of constraint points.

Move points → arrangements vary → number remains constant

Codim-$d$ constraint → degree-$d$ class ∈ (graded) cohomology ring.

Deformed constraints have the same class.
Cohomology ring structure

The structure of the cohomology ring depends on the structure of the underlying moduli space $X$. The cells of a cell-decomposition of $X$ determine a $\mathbb{Z}$-module basis for $H^*(X, \mathbb{Z})$.

Cohomology class with degree $d$ represented by dual homology class (via Poincaré duality).

$[V \cap W] = [V] \ast [W]$ if $V$ and $W$ intersect transversely.

$[V \cup W] = [V] + [W]$. 
Deformation

Deformed constraints have the same class.

Example: Line $ax + by + cz = 0$ in $\mathbb{P}^2$ represented by $[a : b : c]$.

Line through $P[1 : 0 : 0] \iff a = 0$. Line through $Q[0 : 1 : 0] \iff b = 0$.

Deformation $F(t): a(1 - t) + bt = 0$ so $[P\text{-pencil}] = [Q\text{-pencil}]$. 

\begin{align*}
  t = 0 & \quad P[1 : 0 : 0] \\
  t = 1 & \quad Q[0 : 1 : 0]
\end{align*}
Example: $\mathbb{P}^n$ and Bézout’s Theorem

$\mathbb{P}^n$ has one cell in each dimension $d = 0, \ldots, n$

$$H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}c_0 \oplus \mathbb{Z}c_1 \oplus \cdots \oplus \mathbb{Z}c_n \text{ w/ } c_d \text{ codim-}d \text{ class}$$

$c_0 = [\mathbb{P}^n] = \text{Identity element} \quad c_1^2 = c_2 = [\text{codim-2 plane}]$

$c_1 = [\text{hyperplane}] \quad c_1^n = c_n = [\text{point}]$

$$H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[h]/(h^{n+1}) \text{ w/ } h = c_1 = [\text{hyperplane}].$$

**Deformation:** [degree $d$ hypersurface] = $dh$.

**Bézout:** $[d_1\text{-hyp} \cap d_2\text{-hyp} \cap \cdots \cap d_n\text{-hyp}] = (d_1h) \cdots (d_nh) = d_1 \cdots d_nh^n$. 
Example: Generic Arrangements in $\mathbb{P}^n$

Kunneth formula: $H^*((\mathbb{P}^n)^k, \mathbb{Z}) = \mathbb{Z}[h_1, h_2, \ldots, h_k]/(h_1^{n+1}, \ldots, h_k^{n+1})$.

Question

How many generic arrangements of $k$ hyperplanes go through $d = kn$ points?

$A$ goes through $P \iff H_1(P)H_2(P) \cdots H_k(P) = 0$.

\[
[H_1(P)H_2(P) \cdots H_k(P) = 0] = h_1 + \cdots + h_k
\]

\[
[\{\text{arrs satisfying } kn \text{ point conditions }\}] = (h_1 + \cdots + h_k)^{kn} = \binom{kn}{n, n, \ldots, n}(h_1 h_2 \cdots h_k)^n = \binom{kn}{n, n, \ldots, n} \text{ points/arrs}
\]

so answer is $\binom{kn}{n, n, \ldots, n}/k!$. 
Characteristic Numbers

**Definition**

Given a family of plane curves, the characteristic number $N(p, \ell)$ counts the number of such curves that pass through $p$ points and are tangent to $\ell$ lines in general position.

Let $N_k(p, 2k - p)$ be the nontrivial characteristic numbers for the family of generic arrangements of $k$ lines in $\mathbb{P}^2$.

We can compute these for $k = 3$ and $k = 4$ and use them to solve more general enumerative problems.
The Heinz Problem

Question

How many generic arrangements of 3 lines pass through 3 points and are tangent to 3 lines?

\[ M = \{ (L_1, L_2, L_3, P_{12}, P_{13}, P_{23}) : P_{ij} \in L_i \cap L_j \} \subset (\mathbb{P}^2)^3 \times (\mathbb{P}^2)^3 \]
A Comhology Calculation

\[ M = \{(L_1, L_2, L_3, P_{12}, P_{13}, P_{23}) : P_{ij} \in L_i \cap L_j\} \subset (\mathbb{P}^2)^3 \times (\mathbb{P}^2)^3 \]

\[ H^*((\mathbb{P}^2)^3 \times (\mathbb{P}^2)^3, \mathbb{Z}) = \mathbb{Z}[x_1, x_2, x_3, y_{12}, y_{13}, y_{23}]/(x_1^3, \ldots, y_{23}^3) \]

\[ x_i = [\text{hyperplane (point) condition on } L_i] \]
\[ y_{ij} = [\text{hyperplane (line) condition on } P_{ij}] \]

\[ [P_{ij} \in L_i] = \{ax_0 + by_0 + cz_0 = 0\} = [\{ax_0\}] = x_i + y_{ij} \]

\[ [M] = (x_1 + y_{12})(x_2 + y_{12})(x_1 + y_{13})(x_3 + y_{13})(x_2 + y_{23})(x_3 + y_{23}) \]

Labeled Count = \[ [M](x_1 + x_2 + x_3)^3(y_{12} + y_{13} + y_{23})^3 = 342[pt] \]
Count = \[ N_3(3, 3) = 342/3! = 57. \]
Why is this called the Heinz Problem?

57 Varieties!
Theorem (Paul, Wakefield, T-)

**The characteristic numbers for the family of generic arrangements with 3 lines are:**

<table>
<thead>
<tr>
<th>Points $p$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lines $\ell$</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$N_3(p, \ell)$</td>
<td>15</td>
<td>30</td>
<td>48</td>
<td>57</td>
<td>48</td>
<td>30</td>
<td>15</td>
</tr>
</tbody>
</table>

Why is the table symmetric? → Duality!

Why is the table unimodal?
Consider $N_4(0, 8)$. Then $\binom{8}{2} \binom{6}{2}$ quadruple lines each count as $6!$ labeled arrangements in $M = M_A \cup M'$. 

$$
N_4(0, 8) = \left( [M](y_{12} + \cdots + y_{34})^8 - \binom{8}{2} \binom{6}{2} 6! \right) / 4! \\
= 16,695.
$$

By duality there are 16,695 braid arrangements through 8 points.
Theorem (Paul, Wakefield, T-)

The characteristic numbers for the family of generic arrangements with 4 lines are:

<table>
<thead>
<tr>
<th>Points $p$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lines $\ell$</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$N_4(p, \ell)$</td>
<td>16695</td>
<td>17955</td>
<td>13185</td>
<td>8190</td>
<td>4410</td>
<td>2070</td>
<td>855</td>
<td>315</td>
<td>105</td>
</tr>
</tbody>
</table>

What happens with 5 lines? $\rightarrow$ Excess Intersection – higher dimensional collections of arrangements that contribute a finite number to the degree computation.
We consider arrangements where each hyperplane contains a common linear space, but the hyperplanes are otherwise generic.

**d-coned generic arrangement** $\mathcal{A}$ in $\mathbb{P}^n$: $H \cong \mathbb{P}^{n-(d+1)}$ containing generic arr. $\mathcal{B}$, $\Omega$ linear space of dim. $d$, $\Omega \cap H = \emptyset$, each $H$ in $\mathcal{A}$ is span of hyp in $\mathcal{B}$ with $\Omega$. 
Question

How many 1-coned arrangements in $\mathbb{P}^5$ of 6 hyperplanes pass through $D$ points in general position?

Each hyperplane is determined by its normal vector modulo line $\Omega$, i.e. by a point in $\mathbb{P}(\mathbb{C}^6/\Omega) \cong \mathbb{P}^3$.

$$D = \dim M_A = \dim \mathbb{G}(1, 5) \times (\mathbb{P}^3)^6 = 2 \times 4 + 6 \times 3 = 26.$$ 

$\mathbb{G}(1, 5) =$ Grassmannian of lines in $\mathbb{P}^5$. 

Traves (USNA)
Counting Arrangements
Cohomology of the Grassmannian

Each line in \( \mathbb{P}^5 \) is the span of 2 points in \( \mathbb{C}^6 \).

Cellular decomposition of \( \mathbb{G}(1, 5) \): cells determined by position of leading ones in RREF of spanning matrix:

\[
\sigma_{20} = \left\{ M \in M_{2 \times 6} : M \sim \begin{bmatrix} 1 & * & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \end{bmatrix} \right\} \rightarrow \text{codim} \sigma_{20} = 2
\]

Schubert calculus tells us how to multiply cohomology classes
- Giambelli formula \( \rightarrow \) special classes
- Pieri formula \( \rightarrow \) multiplication of special classes

Duality: \( \sigma_\alpha \sigma_{\alpha'} = [\text{single } \Omega] \)
A cohomology class

\[ Z := \{ (\Omega, H) \in \mathbb{G}(1, 5) \times \mathbb{P}^5 : \Omega \subset H \}. \]

\( H \) must contain 2 independent points on \( \Omega \): so \( \text{codim} \ Z = 2. \)

\[ [Z] = a\sigma_{20} + b\sigma_{11} + c\sigma_{10} h + d\sigma_{00} h^2 \]

Use duality to find coefficients:

\[ [Z] = \sigma_{11} + \sigma_{10} h + \sigma_{00} h^2 \]
# 1-coned arr. of 6 hyperplanes through 26 points in \( \mathbb{P}^5 \)

Pull-back to \( G(1, 5) \times (\mathbb{P}^5)^6 \):

\[
[Z_i] = \sigma_{11} + \sigma_{10} h_i + \sigma_{00} h_i^2
\]

\[
\text{ANS} = (h_1 + h_2 + h_3 + h_4 + h_5 + h_6)^{26} \prod_{i=1}^{6} [Z_i]
\]

Schubert Calculus gives:

\[
\sigma_{00}^2 \sigma_{11}^4 = \sigma_{00} \sigma_{10}^2 \sigma_{11}^3 = 1 \sigma_{44} = 1 \text{[arr]} \quad \sigma_{10}^4 \sigma_{11}^2 = 2 \text{[arr]}.
\]

\[
\text{ANS} = \left[ 1 \binom{6}{2,4} \binom{26}{3,3,5,5,5,5} + 1 \binom{6}{1,2,3} \binom{26}{3,4,4,5,5,5} + 2 \binom{6}{4,2} \binom{26}{4,4,4,4,5,5} \right] / 6!
\]

\[
= 10270301132391300
\]

Catalan numbers
General Result

We have a general expression for the dimension $D$ of $M_A$ for $d$-coned arrangements of $k$ hyperplanes in $\mathbb{P}^n$.

We can also express the number of such arrangements passing through $D$ points using Schubert Calculus.
Parting thoughts

1. Need a better sense of how the intersection lattice of $\mathcal{A}$ affects the dimension of $\mathcal{M}_\mathcal{A}$ and the enumerative counts.

2. Is there a recursive relationship between the enumerative counts for $d$-coned arrangements of $k$ hyperplanes in $\mathbb{P}^n$ passing through $D = \dim \mathcal{M}_\mathcal{A}$ points?

3. Is there a role for quantum cohomology here? If so, are the cohomology rings associative and can we leverage this to get nice generating functions for some class of enumerative problems (c.f. moduli of curves)?

4. This project is in its infancy and there’s room for many others to contribute.