Counting Hyperplane Arrangements

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Enumerative Geometry

How many lines pass through 2 points in the plane?

How many planes pass through 3 points in 3-space?

Note: answer depends on position of the points.
Enumerative Geometry

How many lines pass through 2 points in the plane?

How many planes pass through 3 points in 3-space?

Note: answer depends on position of the points.
How many points lie on the intersection of two distinct lines in $\mathbb{R}^2$?

In $\mathbb{R}^2$ the answer depends on the position of the lines.

Work in projective space $\mathbb{P}^2 := \left\{ \text{nonzero points in } \mathbb{R}^3 \right\} / (x, y, z) \sim (\lambda x, \lambda y, \lambda z)$ for nonzero $\lambda$.

Write $[x : y : z]$ for the equiv. class of nonzero $(x, y, z)$.

Note that $\mathbb{P}^2 = \left\{ [x : y : 1] : (x, y) \in \mathbb{R}^2 \right\} \cup \left\{ [x : y : 0] : [x : y] \in \mathbb{P}^1 \right\}$.

A more complicated context gives a simpler answer.
Hypersurfaces

Polynomials in $x, y, z$ no longer give well-defined functions on $\mathbb{P}^2$:

$$P(x, y, z) \neq P(\lambda x, \lambda y, \lambda z).$$

But the zero set of a homogeneous polynomial makes sense:

$$P(\lambda x, \lambda y, \lambda z) = \lambda^d P(x, y, z)$$

if all terms in $P$ have degree $d$.

Hypersurface: the zero set of a nonconstant homogeneous polynomial.

Theorem (Bézout)

The intersection of $n$ hypersurfaces of degrees $d_1, d_2, \ldots, d_n$ in $\mathbb{P}^n$ consists of $d_1 d_2 \cdots d_n$ points, counted appropriately.

Need to work over $\mathbb{C}$. Need multiplicity.
The basic counting method

Find a suitable parameter space for your objects.

Show that the geometric constraints correspond to the intersection of hypersurfaces, $H_1 \cap \cdots \cap H_n$.

Find the degrees of the hypersurfaces.

Obtain the count by Bézout’s Theorem.
An arrangement $\mathcal{A}$ is a finite collection of hyperplanes in $\mathbb{P}^n$.

A line arrangement
$\mathcal{A} = \{x = 0, y = 0, x + y - z = 0\}$

The braid arrangement
$\mathcal{A}_n = \{x_i = x_j : 0 \leq i < j \leq n\}$
The intersection lattice \( \mathcal{L}(\mathcal{A}) \).

Poset: elements = intersections of hyperplanes in \( \mathcal{A} \).
\( U < V \iff U \) contains \( V \).

Properties of \( \mathcal{A} \) that depend only on the intersection lattice are said to be **combinatorial**.
Let $\mathcal{A}$ be a labeled arrangement with intersection lattice $\mathcal{L}(\mathcal{A})$.

$$M = \{ \text{arrangements } \mathcal{B} : \mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{A}) \}.$$

$$\mathcal{A} = \{ H_1 = 0, \ldots, H_k = 0 \} \leftrightarrow (\nabla H_1, \ldots, \nabla H_k) \in (\mathbb{P}^n)^k.$$

$$M(\mathcal{L}(\mathcal{A})) = \overline{M} \text{ in } (\mathbb{P}^n)^k.$$

We’ll deal with unlabeled arrangements later: quotient by $\text{Sym}(k)$.
\[ \mathcal{A} = \{ H_1 = 0, \ldots, H_k = 0 \} \] through \( P \in \mathbb{P}^n \) when

\[ H_1(P)H_2(P) \cdots H_k(P) = 0. \]

\[ \dim \mathbf{M}(\mathcal{L}(\mathcal{A})) = \min \text{ number of point-conditions required to determine a finite number of arrangements through all the points} \]
An open question

Pappus arrangement

dimension appears to be 9 but is actually 10

presence of syzygy affects dimension

How do we “see” a syzygy in the intersection lattice?

Open Question

The dimension of $\mathbf{M}(\mathcal{L}(\mathcal{A}))$ is combinatorial. Find a way to determine the dimension of $\mathbf{M}(\mathcal{L}(\mathcal{A}))$ using only the intersection lattice.
Generic arrangements

Generic in $\mathbb{P}^n$: at most $n$ hyperplanes through any point

**Theorem**

The dimension of $\mathbf{M}(\mathcal{L}(A))$ for $A$ generic with $k$ hyperplanes in $\mathbb{P}^n$ is $nk$ and the number of such arrangements through $nk$ points is

$$\frac{n^k}{k!} = \frac{(nk)!}{(n!)^k(k!)}. $$

Paul, Traves & Wakefield (USNA)

Counting Arrangements

Queen's, 26 JUL 2012

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How many 3-generic line arrangements pass through $p$ points and are tangent to $\ell$ lines ($p + \ell = 6$)?

<table>
<thead>
<tr>
<th>Points $p$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lines $\ell$</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Count $N(p, \ell)$</td>
<td>15</td>
<td>30</td>
<td>48</td>
<td>57</td>
<td>48</td>
<td>30</td>
<td>15</td>
</tr>
</tbody>
</table>

Why is the table symmetric?

**Fulton-Kleiman-MacPherson**: can count 3-generic arrangements through $p$ points and tangent to $\ell = 6 - p$ smooth curves of various degrees using characteristic numbers.
A pencil of lines is a set of lines meeting in a common point.

**Theorem**

If $\mathcal{A}$ is a pencil of $k$ lines in $\mathbb{P}^2$ then $\dim \mathbf{M}(\mathcal{L}(\mathcal{A})) = k + 2$ and the characteristic numbers are as follows:

<table>
<thead>
<tr>
<th>Points $p$</th>
<th>$k + 2$</th>
<th>$k + 1$</th>
<th>$k$</th>
<th>$k - 1$</th>
<th>$\ldots$</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lines $\ell$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>$\ldots$</td>
<td>$k + 2$</td>
</tr>
<tr>
<td>Characteristic $N(p, \ell)$</td>
<td>$3\left(\frac{k + 2}{4}\right)$</td>
<td>$\left(\frac{k + 1}{2}\right)$</td>
<td>1</td>
<td>0</td>
<td>$\ldots$</td>
<td>0</td>
</tr>
</tbody>
</table>
Cohomology

Goal: count arrangements without precise location of constraint points.

Move points $\rightarrow$ arrangements vary $\rightarrow$ number remains constant

Codim-$d$ constraint $\rightarrow$ degree-$d$ class $\in$ (graded) cohomology ring.

Deformed constraints have the same class.
Cohomology ring structure

The structure of the cohomology ring depends on the structure of the underlying moduli space $X$. The cells of a cell-decomposition of $X$ determine a $\mathbb{Z}$-module basis for $H^*(X, \mathbb{Z})$.

$$[V \cap W] = [V] \ast [W] \text{ if } V \text{ and } W \text{ intersect transversely.}$$

$$[V \cup W] = [V] + [W].$$
Deformation

Deformed constraints have the same class.
Example: Line $ax + by + cz = 0$ in $\mathbb{P}^2$ represented by $[a : b : c]$.

Line through $P[1 : 0 : 0] \iff a = 0$. Line through $Q[0 : 1 : 0] \iff b = 0$.

Deformation $F(t): a(1 - t) + bt = 0$ so $[P\text{-pencil}] = [Q\text{-pencil}]$. 

![Diagram](image-url)
Example: $\mathbb{P}^n$ and Bézout’s Theorem

$\mathbb{P}^n$ has one cell in each dimension $d = 0, \ldots, n$

$H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}c_0 \oplus \mathbb{Z}c_1 \oplus \cdots \oplus \mathbb{Z}c_n$ w/ $c_d$ codim-$d$ class

$c_0 = [\mathbb{P}^n] =$ Identity element

$c_1 =$ [hyperplane]

$c_2 = c_2 =$ [codim-2 plane]

$c^n = c_n =$ [point]

$H^*(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}[h]/(h^{n+1})$ w/ $h = c_1 =$ [hyperplane].

Deformation: [degree $d$ hypersurface] = $dh$.

Bézout: $[d_1\text{-hyp} \cap d_2\text{-hyp} \cap \cdots \cap d_n\text{-hyp}] = (d_1h) \cdots (d_nh) = d_1 \cdots d_nh^n$. 
Example: \((\mathbb{P}^n)^k\) and 3-pencils

Kunneth formula: \(H^*((\mathbb{P}^n)^k, \mathbb{Z}) = \mathbb{Z}[h_1, h_2, \ldots, h_k]/(h_1^{n+1}, \ldots, h_k^{n+1})\).

Question

How many 3-pencils in \(\mathbb{P}^2\) go through 5 points?

\[\text{M}(\mathcal{L}(\mathcal{A})) \subset (\mathbb{P}^2)^3\] and \(\mathcal{A}\) goes through \(P\) \(\iff\) \(H_1(P)H_2(P)H_3(P) = 0\).

\[[H_1(P)H_2(P)H_3(P) = 0] = h_1 + h_2 + h_3\]

[3 lines lie in a pencil] = [\(\det(\nabla H_1, \nabla H_2, \nabla H_3) = 0\)] = \(h_1 + h_2 + h_3\).

So \((h_1 + h_2 + h_3)^5(h_1 + h_2 + h_3) = \binom{6}{2,2,2}(h_1 h_2 h_3)^2\)

so answer is \(\binom{6}{2,2,2}/3! = 15\).
Determine a method to find the cohomology class of $M(L(A))$ using only the intersection lattice $L(A)$. 
Example: the 4-pencil \[1\]

Question

*How many 4-pencils in $\mathbb{P}^2$ pass through 6 points?*

\[
\text{det}[123] = \text{det}[124] = \text{det}[134] = \text{det}[234] = 0 \text{ and 6 point conditions}
\]

10 conditions but $M(\mathcal{L}(A)) \subset (\mathbb{P}^2)^4$, of dim 8 $\Rightarrow$ ANS = 0.
6 point conditions and $\det[123]=0$ and $\det[124]=0$ give

$$(h_1 + h_2 + h_3)(h_1 + h_2 + h_4)(h_1 + h_2 + h_3 + h_4)^6 = 1440(h_1 h_2 h_3 h_4)^2$$

So ANS = $1440/4! = 60$
6 point conditions and \( \det[123]=0 \) and \( \det[124]=0 \) give

\[
(h_1 + h_2 + h_3)(h_1 + h_2 + h_4)(h_1 + h_2 + h_3 + h_4)^6 = 1440(h_1 h_2 h_3 h_4)^2
\]

So ANS = \( 1440/4! = 60 \) ??
Example: 4-pencil [3] - Adding parameters

Add a parameter to account for the pencil’s vertex

\[ \mathbf{M}(\mathcal{L}(\mathcal{A}))' = \{(P, H_1, H_2, H_3, H_4) \in \mathbb{P}^2 \times (\mathbb{P}^2)^4 : P \in H_1 \cap H_2 \cap H_3 \cap H_4 \} \]

\[ [P \in H_1] = [H_1(P) = 0] = h_p + h_1. \]

\[ (h_p + h_1)(h_p+h_2)(h_p+h_3)(h_p+h_4)(h_1+h_2+h_3+h_4)^6 = 1080(h_p h_1 h_2 h_3 h_4)^2 \]

ANS = 1080/4! = 45.
Example: the braid arrangement

The braid arrangement in $\mathbb{P}^3$:
\[ \{ x_0 = x_1, x_0 = x_2, x_0 = x_3, x_1 = x_2, x_1 = x_3, x_2 = x_3 \}. \]

Traditional to mod out line $x_0 = x_1 = x_2 = x_3$ to get a line arrangement with $\dim M(\mathcal{L}(A)) = 8$. 
How many braid arrangements go through 8 points?

Four triple points: \( \text{det}[135] = \text{det}[146] = \text{det}[236] = \text{det}[245] = 0 \)

\[
(h_1 + h_3 + h_5)(h_1 + h_4 + h_6)(h_2 + h_3 + h_6)(h_2 + h_4 + h_5)(h_1 + \cdots + h_6)^8 / 4! = 22,995(h_1 \cdots h_6)^2
\]

\{ \text{Dets} = 0 \} = \text{Braids} \cup \text{Pencils}

Need to remove ordered 6-pencils before division:

\[
(6!) \times 3 \binom{8}{4} = 151,200.
\]

Get 400,680 ordered braid arrangements \( \Rightarrow 16,695 \) braids.
Point-line duality in $\mathbb{P}^2$: Line $ax + by + cz = 0 \leftrightarrow$ Point $[a : b : c]$.

\[ P = [x_0 : y_0 : z_0] \quad \text{on line} \quad L : ax + by + cz = 0 \]
\[ \hat{P} : x_0x + y_0y + z_0z = 0 \quad \text{through} \quad \hat{L} = [a : b : c]. \]

Dual of braid is 4-generic: there are 16,695 4-generics through 8 points.
Problem

Count arrangements of 6 hyperplanes in \( \mathbb{P}^5 \) that contain a common line and pass through 26 general points.

\[
\mathbf{M}(\mathcal{L}(\mathcal{A})) \subset \mathbb{G}(1, 5) \times (\mathbb{P}^5)^6
\]

\( \mathbb{G}(1, 5) \): Grassmannian of 1-planes in \( \mathbb{P}^5 \)

\[
\begin{bmatrix}
* & * & * & * & 1 & 0 \\
* & * & * & * & 0 & 1
\end{bmatrix} \rightarrow \dim \mathbb{G}(1, 5) = 8.
\]

Require \( \dim \mathbf{M}(\mathcal{L}(\mathcal{A})) = 8 + 6(3) = 26 \) points.
Cellular decomposition of $\mathbb{G}(1, 5)$: cells determined by position of leading ones:

$$\sigma_{14} = \left\{ M \in M_{2 \times 6} : M \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 1 & 0 & 0 \end{bmatrix} \right\} \rightarrow \dim \sigma_{14} = 2$$

Schubert calculus tells us how to multiply cohomology classes

- Giambelli formula $\rightarrow$ special classes
- Pieri formula $\rightarrow$ multiplication of special classes

Gatto: implemented Schubert calculus using HS-derivations
A cohomology class

\[ Z := \{ (\Lambda, H) \in G(1, 5) \times \mathbb{P}^5 : \Lambda \subset H \} . \]

\( H \) must contain 2 independent points on \( \Lambda \): so \( \text{codim } Z = 2. \)

\[ [Z] = a\sigma_{36} + b\sigma_{45} + c\sigma_{46}h + d\sigma_{56}h^2 \]

Multiply by \( \text{codim}-11 \) classes to find coefficients:

\[ [Z_i] = \sigma_{45} + \sigma_{46}h_i + \sigma_{56}h_i^2 \]
Count 6-pencils of dim 1 through 26 points in $\mathbb{P}^5$

$$[Z_i] = \sigma_{45} + \sigma_{46}h_i + \sigma_{56}h_i^2$$

$$\text{ANS} = (h_1 + h_2 + h_3 + h_4 + h_5 + h_6)^{26} \prod_{i=1}^{6} [Z_i]$$

Schubert Calculus gives:

$$\sigma_{56}^2 \sigma_{45}^4 = \sigma_{56}^2 \sigma_{46}^2 \sigma_{45}^3 = 1 \sigma_{56}.$$  

But

$$\sigma_{46}^4 \sigma_{45}^2 = 2 \sigma_{56}.$$  

$$\text{ANS} = \left[ \binom{6}{2,4} \binom{26}{3,3,5,5,5,5} + \binom{6}{1,2,3} \binom{26}{3,4,4,5,5,5} + 2 \binom{6}{4,2} \binom{26}{4,4,4,4,5,5} \right] / 6! = 10, 270, 301, 132, 391, 300$$
Parting thoughts

1. Need a better sense of how the intersection lattice of $\mathcal{A}$ affects the dimension of $M(\mathcal{L}(\mathcal{A}))$ and the enumerative counts.

2. Currently working to generalize the higher-dimensional example to a theorem about pencils.

3. Is there a way to encode certain enumerative counts into an interesting generating function?

4. Our project is in its infancy and there’s room for many others to contribute.