Lesson 5. Work Scheduling Models

Example 1. Postal employees in Simplexville work for 5 consecutive days, followed by 2 days off, repeated weekly. Below are the minimum number of employees needed for each day of the week:

<table>
<thead>
<tr>
<th>Day</th>
<th>Employees needed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday (1)</td>
<td>7</td>
</tr>
<tr>
<td>Tuesday (2)</td>
<td>8</td>
</tr>
<tr>
<td>Wednesday (3)</td>
<td>7</td>
</tr>
<tr>
<td>Thursday (4)</td>
<td>6</td>
</tr>
<tr>
<td>Friday (5)</td>
<td>6</td>
</tr>
<tr>
<td>Saturday (6)</td>
<td>4</td>
</tr>
<tr>
<td>Sunday (7)</td>
<td>5</td>
</tr>
</tbody>
</table>

Write a linear program that determines the minimum total number of employees needed. You may assume that fractional solutions are acceptable.

Variables:
- \( x_1 \): #employees working on day 1
- \( x_2 \): #employees working on day 2
- \( x_3 \): #employees working on day 3
- \( x_4 \): #employees working on day 4
- \( x_5 \): #employees working on day 5
- \( x_6 \): #employees working on day 6
- \( x_7 \): #employees working on day 7

Objective:
\[
\text{min } y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7
\]

Constraints:
- \( y_1 \): #employees who start on day 1 and end on day 5
- \( y_2 \): #employees who start on day 2 and end on day 6
- \( y_7 \): #employees who start on day 7 and end on day 4

\[
\begin{align*}
\text{which employees have a shift that includes Monday?} \\
\begin{align*}
y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 & \geq 7 \quad (\text{Mon}) \\
y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 & \geq 8 \quad (\text{Tue}) \\
y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 & \geq 7 \quad (\text{Wed}) \\
y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 & \geq 6 \quad (\text{Thu}) \\
y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 & \geq 6 \quad (\text{Fri}) \\
y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 & \geq 5 \quad (\text{Sat}) \\
y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 & \geq 5 \quad (\text{Sun}) \\
y_1 \geq 0, y_2 \geq 0, \ldots, y_7 \geq 0 \quad (\text{nonnegativity})
\end{align*}
\]
Example 2. At Melanie's Kitchen, tables are set and cleared by runners working 4-hour shifts that start on the hour, from 5am to 10am. For example, the shift that starts at 9am ends at 1pm. Melanie's pays $7 per hour for the shifts that start at 5am, 6am, and 7am, and $6 per hour for the shifts that start at 8am, 9am, and 10am. Past experience indicates that the following number of runners are needed at each hour of operation:

<table>
<thead>
<tr>
<th>Hour</th>
<th>Number of runners required</th>
</tr>
</thead>
<tbody>
<tr>
<td>5am-6am</td>
<td>2</td>
</tr>
<tr>
<td>6am-7am</td>
<td>3</td>
</tr>
<tr>
<td>7am-8am</td>
<td>5</td>
</tr>
<tr>
<td>8am-9am</td>
<td>5</td>
</tr>
<tr>
<td>9am-10am</td>
<td>4</td>
</tr>
<tr>
<td>10am-11am</td>
<td>3</td>
</tr>
<tr>
<td>11am-12pm</td>
<td>2</td>
</tr>
<tr>
<td>12pm-1pm</td>
<td>5</td>
</tr>
<tr>
<td>1pm-2pm</td>
<td>6</td>
</tr>
</tbody>
</table>

Formulate a linear program that determines a cost-minimizing staffing plan. You may assume that fractional solutions are acceptable.

\[
\begin{align*}
\text{DV}_5: \quad & x_5 = \text{# runners starting at 5am and ending at 9am} \\
& x_6 = \quad \text{# runners starting at 6am and ending at 10am} \\
& \vdots \\
& x_{10} = \text{# runners starting at 10am and ending at 2pm} \\
\text{min:} \quad & 28(x_5 + x_6 + x_7) + 24(x_8 + x_9 + x_{10}) \\
\text{s.t.} \quad & x_5 \geq 2 \quad (5-6) \\
& x_5 + x_6 \geq 3 \quad (5-7) \\
& x_5 + x_6 + x_7 \geq 5 \quad (5-8) \\
& x_5 + x_6 + x_7 + x_8 \geq 5 \quad (7-8) \\
& x_6 + x_7 + x_8 + x_9 \geq 4 \quad (8-9) \\
& x_7 + x_8 + x_9 + x_{10} \geq 3 \quad (9-10) \\
& x_8 + x_9 + x_{10} \geq 2 \quad (10-11) \\
& x_9 + x_{10} \geq 5 \quad (11-12) \\
& x_{10} \geq 6 \quad (12-1) \\
& x_5 \geq 0, \ x_6 \geq 0, \ldots, \ x_{10} \geq 0 \quad (nonnegativity)
\end{align*}
\]