Lesson 30. Weak and Strong Duality

1 Practice taking duals!

Example 1. State the dual of the following linear programs.

a. minimize $5x_1 + x_2 - 4x_3$
   subject to $x_1 + x_2 + x_3 + x_4 = 19$
   $4x_2 + 8x_4 \leq 55$
   $x_1 + 6x_2 - x_3 \geq 7$
   $x_1$ free, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \leq 0$

b. maximize $19y_1 + 4y_2 - 8z_2$
   subject to $11y_1 + y_2 + z_1 = 15$
   $z_1 + 5z_2 \leq 0$
   $y_1 - y_2 + z_2 \geq 4$
   $x_1 \geq 0$, $y_2 \geq 0$, $z_1$ free, $z_2$ free
2 Weak duality

• Consider the following primal-dual pair of LPs

\[
\begin{align*}
\text{[P]} & \quad \text{maximize} & & c^\top x \\
& \quad \text{subject to} & & Ax \leq b, \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{[D]} & \quad \text{minimize} & & b^\top y \\
& \quad \text{subject to} & & A^\top y \geq c, \quad y \geq 0
\end{align*}
\]

• Remember we constructed the dual in such a way that the multipliers \( y \) give us an upper bound on the optimal value of \( [P] \)

**Weak Duality Theorem.** Let \( x^* \) be a feasible solution to \( [P] \), and let \( y^* \) be a feasible solution to \( [D] \). Then

\[ c^\top x^* \leq b^\top y^* \]

\[ \begin{array}{c}
\text{Proof:} \\
\text{\( c^\top x^* = (A^\top y^*)^\top x^* = y^\top A x^* \leq y^\top b \)}
\end{array} \]

**Corollary 1.** If \( x^* \) is a feasible solution to \( [P] \), \( y^* \) is a feasible solution to \( [D] \), and

\[ c^\top x^* = b^\top y^* \]

then (i) \( x^* \) is an optimal solution to \( [P] \) and (ii) \( y^* \) is an optimal solution to \( [D] \).

\[ \begin{array}{c}
\text{Proof:} \\
\text{(i) \( c^\top x^* = b^\top y^* \geq c^\top x \) for all feasible solutions \( x \) to \( [P] \)}
\end{array} \]

\[ \text{Weak duality} \]

\[ \begin{array}{c}
\text{(ii) \( b^\top y^* = c^\top x^* \leq b^\top y \) for all feasible solutions \( y \) to \( [D] \)}
\end{array} \]

\[ \text{Weak duality} \]
Corollary 2. If \([P]\) is unbounded, then \([D]\) must be infeasible.

\textbf{Proof.} By contradiction. Suppose \([D]\) is feasible.

Let \(\mathbf{y}^*\) be a feasible solution to \([D]\).

Weak duality \(\Rightarrow \mathbf{c}^T \mathbf{x} \leq \mathbf{b}^T \mathbf{y}^*\) for all feasible solutions \(\mathbf{x}\) to \([P]\)

\(\Rightarrow [P]\) cannot be unbounded, which is a contradiction.

Corollary 3. If \([D]\) is unbounded, then \([P]\) must be infeasible.

\textit{Proof.} Similar to the previous corollary.

- Note that primal infeasibility does not imply dual unboundedness
- It is possible that both primal and dual LPs are infeasible
  - See Rader p. 328 for an example
- All these theorems and corollaries apply to arbitrary primal-dual LP pairs, not just \([P]\) and \([D]\) above

3 Strong duality

\textbf{Strong Duality Theorem.} Let \([P]\) denote a primal LP and \([D]\) its dual.

a. If \([P]\) has a finite optimal solution, then \([D]\) also has a finite optimal solution with \underline{the same objective function value}.

b. If \([P]\) and \([D]\) both have feasible solutions, then

- \([P]\) has a finite optimal solution \(\mathbf{x}^*\);
- \([D]\) has a finite optimal solution \(\mathbf{y}^*\);
- the optimal values of \([P]\) and \([D]\) are equal.

- This is an \underline{AMAZING} fact
- Useful from theoretical, algorithmic, and modeling perspectives
- Even the simplex method implicitly uses duality: the reduced costs are essentially dual solutions that are infeasible until the last step