Lesson 23. Degeneracy, Convergence, Multiple Optimal Solutions

0 Warm up

Example 1. Suppose we are using the simplex method to solve the following canonical form LP:

maximize \( 10x + 3y \)
subject to \( x + y + s_1 = 4 \) \hspace{1cm} (1)
\( 5x + 2y + s_2 = 11 \) \hspace{1cm} (2)
\( y + s_3 = 4 \) \hspace{1cm} (3)
\( x \geq 0 \) \hspace{1cm} (4)
\( y \geq 0 \) \hspace{1cm} (5)
\( s_1 \geq 0 \) \hspace{1cm} (6)
\( s_2 \geq 0 \) \hspace{1cm} (7)
\( s_3 \geq 0 \) \hspace{1cm} (8)

Let \( x = (x, y, s_1, s_2, s_3) \). Our current BFS is \( x^t = (0, 4, 0, 3, 0) \) with basis \( B^t = \{ y, s_1, s_2 \} \). The simplex directions are \( d^x = (1, 0, -1, -5, 0) \) and \( d^s = (0, -1, 1, 2, 1) \). Compute \( x^{t+1} \) and \( B^{t+1} \).

- In the above example, the step size \( \lambda_{\text{max}} = 0 \)
- As a result, \( x^{t+1} = x^t \): it looks like our solution didn’t change!
- The basis did change, however: \( B^{t+1} \neq B^t \)
- Why did this happen?
1 Degeneracy

- A BFS $x$ of an LP with $n$ decision variables is **degenerate** if there are **more** than $n$ constraints active at $x$
  - i.e. there are multiple collections of $n$ linearly independent constraints that define the same $x$

**Example 2.** Is $x'$ in Example 1 degenerate? Why?

- In $x' = (0, 4, 0, 3, 0)$ in Example 1, “too many” of the nonnegativity constraints are active
  - As a result, some of the **basic** variables are equal to zero

- Recall: a BFS of a canonical form LP with $n$ decision variables and $m$ equality constraints has
  - **basic** variables, potentially zero or nonzero
  - **nonbasic** variables, always equal to 0

- Suppose $x$ is a degenerate BFS, with $n + k$ active constraints ($k \geq 1$)

- Then **nonnegativity bounds** must be active, which is larger than $n - m$

- Therefore: a BFS $x$ of a canonical form LP is degenerate if

- As a result, a degenerate BFS may correspond to several bases
  - e.g. in Example 1, the BFS $(0, 4, 0, 3, 0)$ has bases:

- Every step of the simplex method
  - does not necessarily move to a geometrically adjacent extreme point
  - does move to an adjacent BFS (in particular, the bases differ by exactly 1 variable)

- At a degenerate BFS, the simplex method might “get stuck” for a few steps
  - Same BFS, different bases, different simplex directions
  - Zero-length moves: $\lambda_{\text{max}} = 0$

- When $\lambda_{\text{max}} = 0$, just proceed as usual

- Simplex computations will normally escape a sequence of zero-length moves and move away from the current BFS
2 Convergence

- In extreme cases, degeneracy can cause the simplex method to cycle over a set of bases that all represent the same extreme point
  - See Rader p. 291 for an example
- Can we guarantee that the simplex method terminates?
- Yes! Anticycling rules exist
- Easy anticycling rule: Bland's rule
  - Fix an ordering of the decision variables and rename them so that they have a common index
  - e.g. \((x, y, s_1, s_2, s_3) \rightarrow (x_1, x_2, x_3, x_4, x_5)\)
  - Entering variable: choose nonbasic variable with smallest index among those corresponding to improving simplex directions
  - Leaving variable: choose basic variable with smallest index among those that define \(\lambda_{\text{max}}\)
3 Multiple optimal solutions

- Suppose our current BFS is $x^t$, and $y$ is the entering variable
- The change in objective function value from $x^t$ to $x^t + \lambda d^y$ ($\lambda \geq 0$) is $\Rightarrow$ We can use reduced costs to compute changes in objective function

- Suppose we solve a canonical form maximization LP with decision variables $x = (x_1, x_2, x_3, x_4, x_5)$ using the simplex method, and end up with:

$$
x^t = (0, 150, 0, 200, 50) \quad B^t = \{x_2, x_3, x_5\}
$$

$$
d_{x_1} = \left(1, -\frac{1}{2}, 0, -\frac{3}{2}, -\frac{1}{2}\right) \quad d_{x_3} = \left(0, -\frac{1}{2}, 1, \frac{1}{2}, 1\right)
$$

$$
\bar{c}_{x_1} = 0 \quad \bar{c}_{x_3} = -25
$$

- Is $x^t$ optimal?

- Are there multiple optimal solutions?
  - Because the reduced cost $\bar{c}_{x_1} = 0$,
  - Let's explore using $x_1$ as an entering variable:

- In general, if there is a reduced cost equal to 0 at an optimal solution, there may be other optimal solutions
  - The zero reduced cost must correspond to a simplex direction with $\lambda_{\text{max}} > 0$