Lesson 8. Cramer’s Rule, Applications to Economic Models

0 Warm up

Example 1. Find the following determinants:

\[
\begin{vmatrix}
2 & 3 & 0 \\
0 & 4 & 5 \\
6 & 0 & 7 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
8 & 3 & 0 \\
3 & 4 & 5 \\
-1 & 0 & 7 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
2 & 8 & 0 \\
0 & 3 & 5 \\
6 & -1 & 7 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
2 & 3 & 8 \\
0 & 4 & 3 \\
6 & 0 & -1 \\
\end{vmatrix}
\]
1 Cramer’s rule

- Suppose we want to solve a system of equations $Ax = d$ for $x$, where $A$ is $n \times n$ and $d$ is $n \times 1$
  - Quick check: $x$ has dimension __________

- Let $A_j$ be the matrix $A$, but with the $j$th column replaced by $d$

- Cramer’s rule:

Example 2. Solve the following system of equations using Cramer’s rule:

\[
\begin{align*}
2x_1 + 3x_2 &= 8 \\
4x_2 + 5x_3 &= 3 \\
6x_1 + 7x_3 &= -1
\end{align*}
\]
2 Two commodity partial market equilibrium

- Market with two products that are related to each other
- Variables:
  
  \[ Q_{d1} \] = quantity demanded for product 1  
  \[ Q_{d2} \] = quantity demanded for product 2  
  \[ Q_{s1} \] = quantity supplied for product 1  
  \[ Q_{s2} \] = quantity supplied for product 2  
  \[ P_1 \] = price of product 1  
  \[ P_2 \] = price of product 2

- A general model with 6 variables and 6 equations:
  
  \[ Q_{d1} = Q_{s1} \]  
  \[ Q_{d1} = a_0 + a_1 P_1 + a_2 P_2 \]  
  \[ Q_{d2} = \alpha_0 + \alpha_1 P_1 + \alpha_2 P_2 \]  
  \[ Q_{s1} = b_0 + b_1 P_1 + b_2 P_2 \]  
  \[ Q_{s2} = \beta_0 + \beta_1 P_1 + \beta_2 P_2 \]

- Depending on the economic context, the parameters \(a_0, a_1, a_2, b_0, b_1, b_2, \alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2\) will have particular signs, magnitudes or relationships between each other
  
  - Product 1 and product 2 are **substitutes** if:
  
  - Product 1 and product 2 are **complements** if:

- Using the equilibrium conditions, we can simplify the above model into 2 variables and 2 equations:

  \[
  (a_1 - b_1)P_1 + (a_2 - b_2)P_2 = -(a_0 - b_0)
  \]

  \[
  (\alpha_1 - \beta_1)P_1 + (\alpha_2 - \beta_2)P_2 = -(\alpha_0 - \beta_0)
  \]

- Using Cramer’s rule, we can find the equilibrium prices:

- Using this closed form solution, we can analytically determine the effects of the parameters on the equilibrium prices
3 A national income model

- Variables:
  
  \[ Y = \text{national income} \quad C = \text{(planned) consumption expenditure} \]

- Parameters:
  
  \[ I_0 = \text{investment expenditures} \quad G_0 = \text{government expenditures} \]
  
  \[ a = \text{autonomous consumption expenditure} \quad b = \text{marginal propensity to consume} \]

- Model:
  
  \[ Y = C + I_0 + G_0 \quad (a > 0, 0 < b < 1) \]
  
  \[ C = a + bY \]

- How is consumption related to national income in this model?

Example 3.

a. Rewrite the national income model above in matrix form, listing the variables in the order \( Y, C \).

b. Solve for variables \( Y \) and \( C \) using Cramer’s rule.