Example 2. Find the local optima of \( f(x_1, x_2, x_3) = 2x_1^2 + 4x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + 2 \). Is \( f \) strictly convex or concave? If so, what can you conclude about the local optima that you found?

- First, we use the first-order necessary condition to find critical points. The gradient of \( f \) is

\[
\nabla f(x_1, x_2, x_3) = \begin{bmatrix} 4x_1 + x_2 + x_3 \\ x_1 + 8x_2 \\ x_1 + 2x_3 \end{bmatrix}
\]

Therefore any critical points must satisfy:

\[
\begin{align*}
4x_1 + x_2 + x_3 &= 0 \\
x_1 + 8x_2 &= 0 \\
x_1 + 2x_3 &= 0
\end{align*}
\]

Solving this system of equations, we find 1 critical point: \((x_1, x_2, x_3) = (0, 0, 0)\).

- Next, we use the second-order sufficient condition to classify the critical points we found. The Hessian matrix of \( f \) is

\[
H(x_1, x_2, x_3) = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{bmatrix}
\]

Therefore, the Hessian matrix at \((x_1, x_2, x_3) = (0, 0, 0)\) is

\[
H(0, 0, 0) = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 8 & 0 \\ 1 & 0 & 2 \end{bmatrix}
\]

and the leading principal minors of \( H(0, 0, 0) \) are

\[
|H_1| = 4 > 0 \quad |H_2| = 31 > 0 \quad |H_3| = 54 > 0
\]

So, \( f(0, 0, 0) = 2 \) is a local minimum.

- Finally, we determine whether \( f \) is strictly convex or concave. Note that for all values of \((x_1, x_2, x_3)\), the leading principal minors of \( H(x_1, x_2, x_3) \) are

\[
|H_1| = 4 > 0 \quad |H_2| = 31 > 0 \quad |H_3| = 54 > 0
\]

Therefore, \( f \) is strictly convex, and \( f(0, 0, 0) = 2 \) is in fact an absolute minimum.
Example 3. Find the local optima of \( f(x_1, x_2, x_3) = -x_1^2 - (x_1 + x_2)^2 - (x_1 + x_3)^2 \). Is \( f \) strictly convex or concave? If so, what can you conclude about the local optima that you found?

- First, we use the first-order necessary condition to find critical points. The gradient of \( f \) is

\[
\nabla f(x_1, x_2, x_3) = \begin{bmatrix}
-6x_1 - 2x_2 - 2x_3 \\
-2x_1 - 2x_2 \\
-2x_1 - 2x_3
\end{bmatrix}
\]

Therefore any critical points must satisfy:

\[
\nabla f(x_1, x_2, x_3) = 0 \iff \begin{cases}
-6x_1 - 2x_2 - 2x_3 = 0 \\
-2x_1 - 2x_2 = 0 \\
-2x_1 - 2x_3 = 0
\end{cases}
\]

Solving this system of equations, we find 1 critical point: \((x_1, x_2, x_3) = (0, 0, 0)\).

- Next, we use the second-order sufficient condition to classify the critical points we found. The Hessian matrix of \( f \) is

\[
H(x_1, x_2, x_3) = \begin{bmatrix}
-6 & -2 & -2 \\
-2 & -2 & 0 \\
-2 & 0 & -2
\end{bmatrix}
\]

Therefore, the Hessian matrix at \((x_1, x_2, x_3) = (0, 0, 0)\) is

\[
H(0, 0, 0) = \begin{bmatrix}
-6 & -2 & -2 \\
-2 & -2 & 0 \\
-2 & 0 & -2
\end{bmatrix}
\]

and the leading principal minors of \(H(0, 0, 0)\) are

\[
|H_1| = -6 < 0 \quad |H_2| = 8 > 0 \quad |H_3| = -8 < 0
\]

So, \(f(0, 0, 0) = 0\) is a local minimum.

- Finally, we determine whether \( f \) is strictly convex or concave. Note that for all values of \((x_1, x_2, x_3)\), the leading principal minors of \(H(x_1, x_2, x_3)\) are

\[
|H_1| = -6 < 0 \quad |H_2| = 8 > 0 \quad |H_3| = -8 < 0
\]

Therefore, \( f \) is strictly convex, and \(f(0, 0, 0) = 0\) is in fact an absolute minimum.
Example 4. Find the local optima of \( f(x_1, x_2, x_3) = 2x_1^2 + x_2^2 + x_3^4 - 4x_1x_3 - 2x_2 \). Is \( f \) strictly convex or concave? If so, what can you conclude about the local optima that you found? Hint. What happens when \( x_3 = 0 \)?

- First, we use the first-order necessary condition to find critical points. The gradient of \( f \) is

\[
\nabla f(x_1, x_2, x_3) = \begin{bmatrix}
4x_1 - 4x_3 \\
2x_2 - 2 \\
4x_3^3 - 4x_1
\end{bmatrix}
\]

Therefore any critical points must satisfy:

\[
\nabla f(x_1, x_2, x_3) = 0 \quad \iff \quad 4x_1 - 4x_3 = 0 \quad 2x_2 - 2 = 0 \quad 4x_3^3 - 4x_1 = 0
\]

Solving this system of equations, we find 3 critical points:

\[
(x_1, x_2, x_3) = (0, 1, 0) \quad (x_1, x_2, x_3) = (-1, 1, -1) \quad (x_1, x_2, x_3) = (1, 1, 1).
\]

- Next, we use the second-order sufficient condition to classify the critical points we found. The Hessian matrix of \( f \) is

\[
H(x_1, x_2, x_3) = \begin{bmatrix}
4 & 0 & -1 \\
0 & 2 & 0 \\
-4 & 0 & 12x_2^2
\end{bmatrix}
\]

1. \((x_1, x_2, x_3) = (0, 1, 0)\). The Hessian matrix at \((x_1, x_2, x_3) = (0, 1, 0)\) is

\[
H(0, 1, 0) = \begin{bmatrix}
4 & 0 & -1 \\
0 & 2 & 0 \\
-4 & 0 & 0
\end{bmatrix}
\]

and the leading principal minors of \(H(0, 1, 0)\) are

\[
|H_1| = 4 > 0 \quad |H_2| = 8 > 0 \quad |H_3| = -8 < 0.
\]

Therefore, \((x_1, x_2, x_3) = (0, 1, 0)\) is a saddle point.

2. \((x_1, x_2, x_3) = (-1, 1, -1)\). The Hessian matrix at \((x_1, x_2, x_3) = (-1, 1, -1)\) is

\[
H(-1, 1, -1) = \begin{bmatrix}
4 & 0 & -1 \\
0 & 2 & 0 \\
-4 & 0 & 12
\end{bmatrix}
\]

and the leading principal minors of \(H(-1, 1, -1)\) are

\[
|H_1| = 4 > 0 \quad |H_2| = 8 > 0 \quad |H_3| = 88 > 0.
\]

Therefore, \((x_1, x_2, x_3) = (-1, 1, -1)\) is a local minimum.

3. \((x_1, x_2, x_3) = (1, 1, 1)\). The Hessian matrix at \((x_1, x_2, x_3) = (1, 1, 1)\) is

\[
H(1, 1, 1) = \begin{bmatrix}
4 & 0 & -1 \\
0 & 2 & 0 \\
-4 & 0 & 12
\end{bmatrix}
\]
and the leading principal minors of $H(1,1,1)$ are

$$|H_1| = 4 > 0 \quad |H_2| = 8 > 0 \quad |H_3| = 88 > 0.$$ 

Therefore, $(x_1, x_2, x_3) = (1,1,1)$ is a local minimum.

- Finally, we determine whether $f$ is strictly convex or concave. From the previous step, we see that we do not have $|H_1| > 0, |H_2| > 0$ and $|H_3| > 0$ for all values of $(x_1, x_2, x_3)$: for example, the Hessian matrix at $(x_1, x_2, x_3) = (0,1,0)$ does not satisfy this condition. Therefore, $f$ is not strictly convex.

We also see from the previous step that we do not have $|H_1| < 0, |H_2| > 0$ and $|H_3| < 0$ for all values of $(x_1, x_2, x_3)$: for example, the Hessian matrix at $(x_1, x_2, x_3) = (0,1,0)$ does not satisfy this condition. Therefore, $f$ is not strictly concave.