Lecture on 4-16 to ?  
SA305 Spring 2014

1 Zero-sum 2 Player Games

A zero sum game is where the sum of each players winnings is equal to zero. In every game each play has certain choices/moves/options/plays to make to try and win. Which order and how often a player uses these different moves is called a players strategy. We have two players call them $P_1$ and $P_2$. Suppose that $P_1$ has $m$ different moves to make and $P_2$ has $n$. Then depending on which move each player makes we can represent $P_1$’s winnings from those choices. So, if $P_1$ chooses $i \in \{1, \ldots, m\}$ as their move and $P_2$ decides to do move $j \in \{1, \ldots, n\}$ then let

$$a_{ij} = P_1’s$$ winnings given $P_1$’s $i$ move and $P_2$’s $j$ move.

The $m \times n$-matrix

$$A = (a_{ij})$$

is called a payoff matrix. (Note: some texts use $a_{ij} =$ how much $P_1$ looses.)

Now suppose that $P_1$ decides to do move $i$ with probability $y_i$ and $P_2$ does move $j$ with probability $x_j$. Then the expected winnings of player $P_1$ is

$$\vec{y}^\top A\vec{x}.$$

If we were to fix a strategy for $P_2$, the column player, then $P_1$ wants to find a strategy that maximizes her expected winnings

$$\max_{\vec{y}} \vec{y}^\top A\vec{x}.$$

However $P_2$ can decide a different strategy and wants to minimize what $P_1$ is maximizing

$$\min_{\vec{x}} \max_{\vec{y}} \vec{y}^\top A\vec{x}.$$

This is the problem that the column player $P_2$ wants to solve. The constraints are that

$$\sum_{i=1}^{n} x_i = 1$$

and $\vec{x} \geq 0$ since $\vec{x}$ is a probability vector.

Similarly, the row player $P_1$ wants to solve the problem

$$\max_{\vec{y}} \min_{\vec{x}} \vec{y}^\top A\vec{x}$$

where the constraints are $\sum_{i=1}^{n} y_i = 1$ and $\vec{y} \geq 0$. These problems do not immediately look like linear programs, but actually we can convert them into linear programs.
First we introduce a new variable. Suppose that
\[ v = \max_{\vec{y}} \vec{y}^\top A\vec{x}. \]

Now \( A\vec{x} \) is a vector that when multiplied with \( \vec{y} \) gives the expected payoff for \( P1 \). The definitely \( v \) must be greater than each coordinate of \( A\vec{x} \) because we could just choose
\[
\vec{y} = e_i = \begin{bmatrix} 0 \\ \\
\vdots \\
0 \\
1 \\
0 \\
\vdots \\
0 \\ \\
\end{bmatrix} \quad \text{\( i^{th} \) coordinate}
\]

From this we have that for all \( i \in \{1, \ldots, m\} \)
\[ v \geq e_i^\top A\vec{x}. \]

For the column player \( P2 \) we have show that the optimal strategy to the minimax problem is actually the solution to the linear program
\[
\begin{align*}
\min & \quad v \\
\text{s.t.} & \quad v \geq e_i^\top A\vec{x} \text{ for all } i \in \{1, \ldots, m\} \\
& \quad \sum_{i=1}^{n} x_i = 1 \\
& \quad \vec{x} \geq 0.
\end{align*}
\]

To turn this into matrix form we note that all the constraints \( v \geq e_i^\top A\vec{x} \) can be written as
\[ -A\vec{x} + ve \geq \vec{0} \]

where \( e \) is the all ones vector
\[
e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}
\]
of length \( m \). Writing the variables in vector form
\[
\begin{bmatrix} \vec{x} \\ v \end{bmatrix}
\]
we can now put the entire linear program in matrix form.
\[
\begin{align*}
&\min \begin{bmatrix} \bar{0} & 1 \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix} \\
&\text{s.t.} \begin{bmatrix} -A & e \\ e^\top & 0 \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix} \geq \begin{bmatrix} \bar{0} \\ 1 \end{bmatrix} \\
&\bar{x} \geq 0, \ v\text{-unrestricted}.
\end{align*}
\]

Examining the row players P1 maximin problem in the same way we let

\[ u = \min_{\bar{x}} \bar{y}^\top A \bar{x}. \]

Then with the same notation we have

\[ u \leq \bar{y}^\top A e_i. \]

Hence the linear program we get in this case is

\[
\begin{align*}
&\max \quad u \\
&\text{s.t.} \quad u \leq \bar{y}^\top A e_i \quad \text{for all } i \in \{1, \ldots, n\} \\
&\quad \sum_{i=1}^n y_i = 1 \\
&\quad \bar{y} \geq 0.
\end{align*}
\]

Multiplying both sides of the constraints from \( A \) again by \( e^\top \) instead of \( e \) we get

\[ u e^\top \leq \bar{y}^\top A. \]

Taking transpose of both sides we have

\[ u e \leq A^\top \bar{y}. \]

Hence in matrix form the rows players linear program is

\[
\begin{align*}
&\max \begin{bmatrix} \bar{0} & 1 \end{bmatrix} \begin{bmatrix} \bar{y} \\ u \end{bmatrix} \\
&\text{s.t.} \begin{bmatrix} -A^\top & e \\ e^\top & 0 \end{bmatrix} \begin{bmatrix} \bar{y} \\ u \end{bmatrix} \leq \begin{bmatrix} \bar{0} \\ 1 \end{bmatrix} \\
&\bar{y} \geq 0, \ u\text{-unrestricted}.
\end{align*}
\]

By a quick inspection these two linear programs are duals of each other. This together with the \textbf{Strong Duality Theorem} proves the famous maximin theorem which was originally proved by John von Neumann and who won the Nobel Prize for this result in economics.

\textbf{Theorem 1.1.} The row players optimal value \( u^* \) is equal to the column players optimal value \( v^* \)

\[ u^* = v^*. \]

Now we examine an example.
2 Poker

The game we are about to study is a very simplified poker game but the solution to the problem gives insights into how optimal play might be conducted in a real setting.

This game is called a limit one half street clairvoyance game. Here are the rules and setting:

- Two players get one card.
- Pot has $P$ dollars.
- Highest card wins and each player has .5 probability of winning.
- $P_1$ can bet 1 dollar or pass.
- $P_2$ can call (match $P_1$’s bet or pass) or fold (only when $P_1$ bets).
- $P_1$ is clairvoyant, can see $P_2$’s cards.

If $P_1$ checks then the only thing that $P_2$ can do is pass and they each win with equal probability. The interesting thing to consider is when $P_1$ bets. There are two cases the one where $P_1$ knows they have won and when they know they have lost. The payoff matrix we study is the following:

<table>
<thead>
<tr>
<th></th>
<th>$P_2$ calls</th>
<th>$P_2$ folds</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1$ not bluff bet</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$P_1$ bluff bet</td>
<td>-1</td>
<td>$P$</td>
</tr>
</tbody>
</table>

Taking this payoff matrix and putting it into the matrix form LP for the $P_1$’s perspective we get:

\[
\begin{align*}
\text{max} & \quad [0 \ 0 \ 1] \begin{bmatrix} \bar{y} \\ u \end{bmatrix} \\
\text{s.t.} & \quad \begin{bmatrix} -1 & 1 & 1 \\ 0 & -P & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ u \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
& \quad \bar{y} \geq 0, \ u\text{-unrestricted.}
\end{align*}
\]

Unfortunately we can not run this on ampl because there is the variable $P$ in the constraint coefficients which makes this NOT linear and ampl runs strictly with numerical examples. But we just want to treat $P$ as a constant and solve the optimization problem for any pot size $P$. The proof of the Strong Duality Theorem gives us the solution since we can construct the dual solution if we know a basis for the optimal solution! But here there are 3 variables and 3 constraints. This means that there will not be any non-basic variables.
and hence the basis $B$ is all the variables. Then the construction in the proof says that the optimal in the dual is (in this case the variables we want are $[\vec{x}, v]$)

$$[\vec{x}, v]^* = \vec{c}_B B^{-1}$$

where in this case

$$B = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -P & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and it’s inverse is a little messy to write out but $\vec{c}_B = [0, 0, 1]$ is the entire objective function coefficient vector. The solution is

$$\vec{x}^* = \begin{bmatrix} \frac{P}{P+2} & \frac{P+2}{P+2} & \frac{P}{P+2} \end{bmatrix}.$$ 

Repeating this same process for $P1$’s strategy we get

$$[\vec{y}, u]^* = \begin{bmatrix} \frac{P+1}{P+2} & \frac{1}{P+2} & \frac{P}{P+2} \end{bmatrix}.$$