1 Bounds on optimal values

Suppose that for some reason we can not solve a linear program via the simplex method. It would be nice to have bounds on the optimum value of the objective function. In this section we discuss how to find some bounds by examining the following example.

Example 1.1. Consider the following linear program in standard form:

\[
\begin{align*}
\text{max} & \quad 3x_1 + x_2 + 4x_3 \\
\text{s.t.} & \quad 2x_1 + x_2 + 2x_3 \leq 10 \\
& \quad \frac{1}{2}x_1 + 3x_2 + x_3 \leq 50 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

Now we could solve this LP via simplex and the optimal solution is \(x_1 = x_2 = 0\) and \(x_3 = 5\) and the optimal value is \(v_{opt} = 20\). But suppose that we didn’t know the value for \(v_{opt}\). Then it would be nice to find numbers \(\ell\) and \(u\) such that

\[\ell \leq v_{opt} \leq u.\]

Of course the closer \(\ell\) and \(u\) are to \(v_{opt}\) the better. Here \(\ell\) is called a lower bound for \(v_{opt}\) and \(u\) is called an upper bound for \(v_{opt}\). Finding a lower bound is actually easier than finding an upper bound for a maximization problem. If you can find a feasible solution, for example, \(\bar{x} = [2, 2, 2]\) then we know that the optimal value has to be greater than or equal to the objective function evaluated on this feasible solution:

\[3 \times 2 + 1 \times 2 + 4 \times 2 = 14 \leq v_{opt}.\]

So we have a lower bound \(\ell = 14\). To find an upper bound is a little more tricky. Here’s the idea:

- Pick a constraint that looks close to the objective function.
- Find a constant \(y\) such that \(y\) multiplied to the entire constraint results in a linear function where each coefficient is bigger than the objective function.
- Then the left hand side of the constraint after being multiplied by \(y\) is an upper bound for \(v_{opt}\).

Let’s do this for the second constraint. Obviously, since all coefficients are positive we could multiply the constraint by 10000000000000 and we would get that all the coefficients are greater than that of the objective function. However, this would result in a gigantic upper bound. But we want a smaller upper bound so that we have more information about the actually value of \(v_{opt}\). So, we want to find the smallest number to multiply this constraint
by so that the coefficients are all greater than or equal to that of the objective function. In this case this number is 6:

\[ 6 \left( \frac{1}{2} x_1 + 3 x_2 + x_3 \right) \leq 6 \times (50). \]

The result is that

\[ 3x_1 + x_2 + 4x_2 \leq 3x_1 + 18x_2 + 6x_3 \]

because of the nonnegativity constraints. Which means that

\[ 3x_1 + x_2 + 4x_2 \leq 3x_1 + 18x_2 + 6x_3 \leq 6 \times 50 = 300. \]

So an upper bound is

\[ v_{opt} \leq u = 300. \]

However this is just using one constraint. What about the other constraint? If we use the first constraint we find a much smaller upper bound

\[ 3x_1 + x_2 + 4x_2 \leq 2(2x_1 + x_2 + 2x_3 \leq 2 \times 10 = 20 = u. \]

Now what if we used these two constraints together? This is the main idea of creating what is called the dual of the linear program. We do this in the next section.

## 2 The dual of a standard form LP

Let’s again look at this same example. We want to find constants so that after multiplying these times each constraint and adding we get a linear function that is just slightly larger than the objective function. So, we want two constants \( y_1 \) and \( y_2 \) corresponding to constraint 1 and 2 respectively such that

\[ y_1 (2x_1 + x_2 + 2x_3) + y_2 (1/2 x_1 + 3x_2 + x_3) \geq 3x_1 + x_2 + 4x_3. \]

Expanding this inequality via the variables we get

\[
\begin{align*}
(2y_1 + 1/2y_2) \cdot x_1 + (y_1 + 3y_2) \cdot x_2 + (2y_1 + y_2) \cdot x_3 \\
\geq 3 \cdot x_1 + 1 \cdot x_2 + 4 \cdot x_3.
\end{align*}
\]

This is basically giving us three constraints

\[
\begin{align*}
2y_1 + 1/2y_2 & \geq 3 \\
y_1 + 3y_2 & \geq 1 \\
2y_1 + y_2 & \geq 4.
\end{align*}
\]
Now we want these inequalities to hold and at the same time have the right hand side of
this combination which is

\[ 10y_1 + 50y_2 \]

to be as small as possible because it is the upper bound that we are looking for. Also note
that the variables \( y_1 \) and \( y_2 \) need to be positive because otherwise if we multiply a constraint
by a negative then the inequality switches direction. So we have constructed the following
LP:

\[
\begin{align*}
\text{min} & \quad 10y_1 + 50y_2 \\
\text{s.t.} & \quad 2y_1 + 1/2y_2 \geq 3 \\
& \quad y_1 + 3y_2 \geq 1 \\
& \quad 2y_1 + y_2 \geq 4 \\
& \quad y_1, y_2 \geq 0.
\end{align*}
\]

This is what is called the dual of our original LP (called the primal). Looking at this
example in matrix form we have

<table>
<thead>
<tr>
<th><strong>Primal LP</strong></th>
<th><strong>Dual LP</strong></th>
</tr>
</thead>
</table>
| \[
\begin{align*}
\text{max} & \quad [3, 1, 4] \vec{x} \\
\text{s.t.} & \quad \begin{bmatrix} 2 & 1 & 2 \\ 1/2 & 3 & 1 \end{bmatrix} \vec{x} \leq \begin{bmatrix} 10 \\ 50 \end{bmatrix} \\
& \quad \vec{x} \geq 0.
\end{align*}
\] | \[
\begin{align*}
\text{min} & \quad [10, 50] \vec{y} \\
\text{s.t.} & \quad \begin{bmatrix} 2 & 1/2 \\ 1 & 3 \\ 2 & 1 \end{bmatrix} \vec{y} \geq \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \\
& \quad \vec{y} \geq 0.
\end{align*}
\] |

This example illustrates exactly what happens in the general case:

<table>
<thead>
<tr>
<th><strong>Primal LP</strong></th>
<th><strong>Dual LP</strong></th>
</tr>
</thead>
</table>
| \[
\begin{align*}
\text{max} & \quad \vec{c}^\top \vec{x} \\
\text{s.t.} & \quad A \vec{x} \leq \vec{b} \\
& \quad \vec{x} \geq 0.
\end{align*}
\] | \[
\begin{align*}
\text{min} & \quad \vec{b}^\top \vec{y} \\
\text{s.t.} & \quad A^\top \vec{y} \geq \vec{c} \\
& \quad \vec{y} \geq 0.
\end{align*}
\] |

**NOTES:**

- \( \vec{x} = [x_1, \ldots, x_n] \) so the number of variables in \( n \).

- \( A \) is a \( m \times n \)-matrix.

- \( \vec{y} = [y_1, \ldots, y_m] \) so the number of variables of the dual is the number of constraints of
  the primal.

- \( A^\top \) is the transpose; so it is a \( n \times m \)-matrix.

- The main idea of the dual is that constraints go to variables and variables go to
  constraints.
Now there are two major questions we need to address:
1) What happens if our linear program is not in canonical form?
and
2) How good is the optimal solution of the dual as an upper bound for the primal?
We answer these questions in the next two sections respectively. First we look at another example.

Example 2.1. Consider the LP in standard form

\[
\begin{align*}
\text{max} & \quad 3x_1 + 3x_2 + 3x_3 \\
\text{s.t.} & \quad 2x_1 + x_2 + 2x_3 \leq 5 \\
& \quad x_1 + 2x_2 + x_3 \leq 6 \\
& \quad x_1, x_2, x_3 \geq 0.
\end{align*}
\]

Looking at each constraint we get upper bounds 15 and 18 respectively. But if we add the two constraints we get the upper bound 11. The dual is

\[
\begin{align*}
\text{min} & \quad 5y_1 + 6y_2 \\
\text{s.t.} & \quad 2y_1 + y_2 \leq 3 \\
& \quad y_1 + 2y_2 \leq 3 \\
& \quad 2y_1 + y_2 \leq 3 \\
& \quad y_1, y_2 \geq 0.
\end{align*}
\]

3 Constructing the dual of an arbitrary LP

There are two ways to construct duals of LP’s not in standard form:
1) convert your LP into standard form
2) construct the dual directly
We describe how to do each. First we look at the way to convert to standard form. Assuming we are working with a maximization problem the usual setting for a constraint is

\[
\sum_{j=1}^{n} a_{ij}x_j \leq b_i
\]

and the usual setting for a variable constraint is \( x_j \geq 0 \). To construct a standard form equivalent LP just follow the rules:

- Keep the usual constraints and variable constraints the same.
• For a constraint of the form
\[ \sum_{j=1}^{n} a_{ij} x_j \geq b_i \]
multiply this by a negative
\[ -\sum_{j=1}^{n} a_{ij} x_j \leq -b_i. \]

• For a constraint of the form
\[ \sum_{j=1}^{n} a_{ij} x_j = b_i \]
make two constraints
\[ \sum_{j=1}^{n} a_{ij} x_j \leq b_i \]
and
\[ -\sum_{j=1}^{n} a_{ij} x_j \leq -b_i. \]

• If \( x_i \leq 0 \) then let \( x_i = -x_i' \). So the new variable is \( x_i' \geq 0 \).

• If \( x_i \) is unrestricted then let \( x_i = x_i^+ - x_i^- \). So, again the new variables satisfy \( x_i^+ \geq 0 \) and \( x_i^- \geq 0 \).

We illustrate with an example.

**Example 3.1.**

\[
\begin{align*}
\text{max} & \quad 3x_1 + 4x_2 + 5x_3 \\
\text{s.t.} & \quad x_1 + x_2 + x_3 \leq 6 \\
& \quad 2x_1 + 3x_2 + 4x_3 \geq 3 \\
& \quad x_1 + x_2 + x_3 = 5 \\
& \quad x_1 \geq 0, x_2 \text{ unrest}, x_3 \leq 0.
\end{align*}
\]

Using the rules we get

\[
\begin{align*}
\text{max} & \quad 3x_1 + 4(x_2^+ - x_2^-) - 5x_3' \\
\text{s.t.} & \quad x_1 + (x_2^+ - x_2^-) - x_3' \leq 6 \\
& \quad -2x_1 - 3(x_2^+ - x_2^-) - 4(-x_3') \leq 3 \\
& \quad x_1 + (x_2^+ - x_2^-) - x_3' \leq 5 \\
& \quad -x_1 - (x_2^+ - x_2^-) + x_3' \leq -5 \\
& \quad x_1, x_2^+, x_2^-, x_3' \geq 0.
\end{align*}
\]

Now we illustrate how to compute the dual directly. Here is the construction for a primal LP P with \( n \) variables and \( m \) constraints.
1. For each non-variable constraint in $P$ create a dual variable $y_i$.

2. The objective function in the dual is the bounds on the non-variable constraints in the primal.

3. The objective function in the primal is the bounds for the non-variable constraints in the dual.

4. The non-variable constraints in the dual is the transpose of the matrix in the primal.

5. The following table gives the direction of the inequalities or equalities in both the non-variable and variable constraints (this is taken almost exactly from the book, page 325-326):

<table>
<thead>
<tr>
<th>MAX problem</th>
<th>↔</th>
<th>MIN Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>(constraint) ≤ #</td>
<td>↔</td>
<td>$y_i ≥ 0$</td>
</tr>
<tr>
<td>(constraint) ≥ #</td>
<td>↔</td>
<td>$y_i ≤ 0$</td>
</tr>
<tr>
<td>(constraint) = #</td>
<td>↔</td>
<td>$y_i$ unrestricted</td>
</tr>
<tr>
<td>$x_i ≥ 0$</td>
<td>↔</td>
<td>(constraint) ≥ #</td>
</tr>
<tr>
<td>$x_i ≤ 0$</td>
<td>↔</td>
<td>(constraint) ≤ #</td>
</tr>
<tr>
<td>$x_i$ unrestricted</td>
<td>↔</td>
<td>(constraint) = #</td>
</tr>
</tbody>
</table>

Again the main idea here is that

CONSTRANTS $\longrightarrow$ VARIABLES

VARIABLES $\longrightarrow$ CONSTRAINTS

One way to remember these is that there are some usual constraints and they go to usual variable constraints and visa versa.

<table>
<thead>
<tr>
<th>Primal Constraints</th>
<th>Dual Variable signs</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAX</td>
<td>MIN</td>
</tr>
<tr>
<td>Sensible</td>
<td>≤</td>
</tr>
<tr>
<td>Odd</td>
<td>=</td>
</tr>
<tr>
<td>Bizarre</td>
<td>≥</td>
</tr>
</tbody>
</table>

4 Duality Theorems

The results in this section are exactly the reason why we are studying the dual. Basically the idea is that the optimal value in the primal is equal to the optimal in the dual. We will exhibit this and the needed hypothesis in the following results. For proof see the lectures in class or the book Section 9.3.
In this entire section we will assume that $P$ represents the primal LP in standard form

$$P = \begin{cases} \max \, c^\top x \\ \text{s.t.} \quad Ax \leq b \\ x \geq 0 \end{cases}$$

and $D$ is its dual

$$D = \begin{cases} \min \, b^\top y \\ \text{s.t.} \quad A^\top y \leq c \\ y \geq 0 \end{cases}$$

**Theorem 4.1** (Weak Duality). If $\bar{x}$ and $\bar{y}$ are feasible in $P$ and $D$ respectively then

$$c^\top \bar{x} \leq b^\top \bar{y}.$$ 

Note that this says that ALL the objective values of feasible solutions in the primal are less than or equal to than that in the dual. We can view this in the following picture where the shaded values on the number line represent the objective function values:

From this picture we can immediately see the following results.

**Corollary 4.2.** If $\bar{x}$ and $\bar{y}$ are feasible in $P$ and $D$ respectively such that

$$c^\top \bar{x} = b^\top \bar{y}$$

then this value is the optimal value for both the primal and the dual

$$v_{opt}(P) = c^\top \bar{x} = b^\top \bar{y} = v_{opt}(D)$$

and hence $\bar{x}$ and $\bar{y}$ are optimal solutions (this value corresponds to the orange point in the middle of the picture).

**Corollary 4.3.** If $P$ in unbounded then $D$ is infeasible.

**Corollary 4.4.** If $D$ in unbounded then $P$ is infeasible.

Now the main result.

**Theorem 4.5** (Strong Duality). If $P$ and $D$ both have feasible solutions then

1. both $P$ and $D$ have finite optimal solutions say $\bar{x}^*$ and $\bar{y}^*$ respectively
2. *the optimal values are the same*

\[ v_{opt}(P) = c^\top \bar{x}^* = \bar{b}^\top \bar{y}^* = v_{opt}(D). \]

**Proof.** The contrapositive of the above Corollaries implies that both \( P \) and \( D \) have finite solutions. Let’s start out assuming we know the optimal solution for \( P \) call it \( \bar{x}^* \). We can also assume that it is a basic feasible solution with basis \( B \), so we can decompose it into its basic and non-basic variables \( \bar{x}^* = [\bar{x}_B : \bar{x}_N] \). Also reorder the constraints of \( P \)

\[ A = [B : N] \]

and the objective function coefficient vector

\[ \tilde{c}^\top = [\tilde{c}_B : \tilde{c}_N]. \]

The optimal value for \( P \) is

\[ v_{opt} = \tilde{c}_B^\top \bar{x}_B = \tilde{c}_B^\top B^{-1} \bar{b}. \]

Let

\[ \bar{y}^\top = \tilde{c}_B^\top B^{-1}. \]

Then the objective function evaluated on \( \bar{y} \) in the dual is

\[ \bar{b}^\top \bar{y} = \bar{y}^\top \bar{b} = \tilde{c}_B^\top B^{-1} \bar{b} = v_{opt} \]

and by the above Corollary this implies that \( \bar{y} \) is an optimal solution for \( D \).

Unfortunately we do not know if \( \bar{y} \) is feasible. We need to show that \( A^\top \bar{y} \geq \tilde{c} \) but in the partition this becomes \( B^\top \bar{y} \geq \tilde{c}_B \) and \( N^\top \bar{y} \geq \tilde{c}_N \). The first inequality there is easy because \( \bar{y}^\top B = \tilde{c}_B^\top \). For the second we need to recall a little more detail of the simplex method. Recall that in finding simplex directions we set and solve \( d_k^j = 1, d_j^k = 0 \) for all \( j \in N \), and \( A d_k^k = \bar{0} \). But this means with the basic decomposition that \( B d_k^B + A_k = \bar{0} \) where \( A_k \) is the \( k^{th} \) column of \( A \). So \( d_B^k = -B^{-1} A_k \) and hence \( -B^{-1} N \) gives all the simplex directions for \( \bar{x}^* \). Also recall the vector of reduced costs

\[ \hat{c}_N = \left( c_k + \sum_{i \in B} c_id_i^k \right)_{k \in N}. \]

So,

\[ \tilde{c}_N^\top - \hat{c}_N^\top = - \left( \sum_{i \in B} c_id_i^k \right)_{k \in N} = \tilde{c}_B^\top B^{-1} N. \]

Hence

\[ \bar{y}^\top = \tilde{c}_B^\top B^{-1} N = \hat{c}_N^\top - \hat{c}_N^\top \geq \hat{c}_N^\top \]

because \( \bar{x}^* \) is an optimal solution for \( P \) which means that all the simplex directions are not improving and that means that the reduced costs vector \( \hat{c}_N^\top \leq \bar{0} \).
This proof shows that the simplex method is much deeper than we originally thought and contains information about the dual. The optimal solution for $D$ 

\[ \vec{y}^\top = \vec{c}_B^\top B^{-1} \]

is basically given by the basic variables of the optimal solution for $P$. Here is the main idea

\[
\begin{array}{c|c}
P & D \\
\hline
\vec{x}^* & \vec{y}^* = \vec{c}_B^\top B^{-1} \\
B & \text{active constraints} \\
N & \text{non-active constraints}
\end{array}
\]

If we track the slack variables in each LP with this correspondence then we get what is called complementary slackness.

**Theorem 4.6 (Complementary Slackness).** Suppose $P$ and $D$ are duals with optimal solutions $\vec{x}^*$ and $\vec{y}^*$ respectively. Let $s_i^*$ be the slack of the $i^{th}$ constraint of $P$ for $\vec{x}^*$ and $w_j^*$ be the surplus of the $j^{th}$ constraint of $D$ for $\vec{y}^*$. Then 

\[ x_j^* w_j^* = 0 \; \forall j \in \{1, \ldots, n\} \]

and 

\[ y_i^* s_i^* = 0 \; \forall i \in \{1, \ldots, m\}. \]