Applications of DEs:
Simple LRC circuits

An LRC circuit is a closed loop containing an inductor of $L$ henries, a resistor of $R$ ohms, a capacitor of $C$ farads, and an EMF (electro-motive force), or battery, of $E(t)$ volts, all connected in series.

They arise in several engineering applications. For example, AM/FM radios with analog tuners typically use an LRC circuit to tune a radio frequency. Most commonly a variable capacitor is attached to the tuning knob, which allows you to change the value of $C$ in the circuit and tune to stations on different frequencies [R].

We use the following “dictionary” to translate between the diagram and the DEs.

<table>
<thead>
<tr>
<th>EE object</th>
<th>term in DE (the voltage drop)</th>
<th>units</th>
<th>symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>charge</td>
<td>$q = \int i(t), dt$</td>
<td>coulombs</td>
<td></td>
</tr>
<tr>
<td>current</td>
<td>$i = q'$</td>
<td>amps</td>
<td></td>
</tr>
<tr>
<td>emf</td>
<td>$e = e(t)$</td>
<td>volts $V$</td>
<td></td>
</tr>
<tr>
<td>resistor</td>
<td>$Rq' = Ri$</td>
<td>ohms $\Omega$</td>
<td></td>
</tr>
<tr>
<td>capacitor</td>
<td>$C^{-1}q$</td>
<td>farads</td>
<td></td>
</tr>
<tr>
<td>inductor</td>
<td>$Lq'' = Li'$</td>
<td>henries</td>
<td></td>
</tr>
</tbody>
</table>

*Kirchoff’s First Law*: The algebraic sum of the currents travelling into any node is zero.

*Kirchoff’s Second Law*: The algebraic sum of the voltage drops around any closed loop is zero.

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Generally, the charge at time $t$ on the capacitor, $q(t)$, satisfies the DE

$$Lq'' + Rq' + \frac{1}{C}q = e(t). \quad (1)$$

When there is no EMF, sometimes the following terminology is used. If $R > 2\sqrt{L/C}$ ("$R$ is large") then the circuit is called **overdamped**. If $R = 2\sqrt{L/C}$ ("$R$ is large") then the circuit is called **critically-damped**. If $0 \geq R < 2\sqrt{L/C}$ ("$R$ is large") then the circuit is called **underdamped**.

**Example 1**: Consider the simple LC circuit given by the diagram in Figure 1.

![Figure 1: A simple LC circuit.](image)

This is a simple model illustrating the idea of a radio tuner (the variable capacitor) which can tune into only two stations, "channel 2" and "channel 11".

According to Kirchoff’s 2\textsuperscript{nd} Law and the above "dictionary", this circuit corresponds to the DE

$$q'' + \frac{1}{C}q = \sin(2t) + \sin(11t).$$

The homogeneous part of the solution is

$$q_h(t) = c_1 \cos(t/\sqrt{C}) + c_1 \sin(t/\sqrt{C}).$$

If $C \neq 1/4$ and $C \neq 1/121$ then

$$q_h(t) = \frac{1}{C-1-4} \sin(2t) + \frac{1}{C-1-121} \sin(11t).$$
When $C = 1/4$ and the initial charge and current are both zero, the solution is

$$q(t) = - \frac{1}{117} \sin(11t) + \frac{161}{936} \sin(2t) - \frac{1}{4} t \cos(2t).$$

This is displayed in Figure 2.
You can see how the frequency $\omega = 2$ dominates the other terms.

When $0 < R < 2\sqrt{L/C}$ the homogeneous form of the charge in (1) has the form

$$q_h(t) = c_1e^{\alpha t}\cos(\beta t) + c_2e^{\alpha t}\sin(\beta t),$$

where $\alpha = -R/2L < 0$ and $\beta = \sqrt{4L/C - R^2/(2L)}$. This is sometimes called the **transient part** of the solution. The remaining terms in the charge are called the **steady state terms**.

**Example:** An LRC circuit has a 1 henry inductor, a 2 ohm resistor, $1/5$ farad capacitor, and an EMF of $50\cos(t)$. If the initial charge and current is 0, since the charge at time $t$.

The IVP describing the charge $q(t)$ is

$$q'' + 2q' + 5q = 50\cos(t), \quad q(0) = q'(0) = 0.$$ 

The homogeneous part of the solution is

$$q_h(t) = c_1e^{-t}\cos(2t) + c_2e^{-t}\sin(2t).$$

The general form of the particular solution using the method of undetermined coefficients is

$$q_p(t) = A_1\cos(t) + A_2\sin(t).$$

Solving for $A_1$ and $A_2$ gives

$$q_p(t) = -10e^{-t}\cos(2t) - \frac{15}{2}e^{-t}\sin(2t).$$

```
SAGE
sage: t = var("t")
sage: q = function("q",t)
sage: L,R,C = var("L,R,C")
sage: E = lambda t: 50*cos(t)
sage: de = lambda y: L*diff(y,t,t) + R*diff(y,t) + (1/C)*y-E(t)
sage: L,R,C = 1,2,1/5
sage: de(q(t))
diff(q(t), t, 2) + 2*diff(q(t), t, 1) + 5*q(t) - 50*cos(t)
sage: desolve_laplace(de(q(t)),["t","q"],[0,0,0])
```
This plot (the solution superimposed with the transient part of the solution) is displayed in Figure 3.

Figure 3: A LRC circuit, with damping, and the transient part (dashed) of the solution.

**Exercise:** Use SAGE to solve

\[ q'' + \frac{1}{C} q = \sin(2t) + \sin(11t), \quad q(0) = q'(0) = 0, \]

in the case \( C = 1/121 \).
References


