Applications of DEs:
Spring problems, III

Prof. Joyner

If the frequency of the driving force of the spring matches the frequency of the homogeneous part $x_h(t)$, in other words if

$$x'' + \omega^2 x = F_0 \cos(\gamma t),$$

satisfies $\omega = \gamma$ then we say that the spring-mass system is in (pure, mechanical) resonance. For some time, it was believed that the collapse of the Tacoma Narrows Bridge [1] was explained by this phenomenon but this is false [2].

Fact: Consider

$$x'' + \omega^2 x = F_0 \cos(\gamma t), \quad (1)$$

and

$$x'' + \omega^2 x = F_0 \sin(\gamma t). \quad (2)$$

Let $x_p = x_p(t)$ denote a particular solution to either (1) or (2). Then we have

- in case (1): if $\gamma \neq \omega$ then

$$x_p(t) = \frac{F_0}{\omega^2 - \gamma^2} \cos(\gamma t),$$

- in case (1): if $\gamma = \omega$ then

$$x_p(t) = \frac{F_0}{2\omega^2} t \sin(\gamma t),$$

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• in case (2): if $\gamma \neq \omega$ then

$$x_p(t) = \frac{F_0}{\omega^2 - \gamma^2} \sin(\gamma t),$$

• in case (2): if $\gamma = \omega$ then

$$x_p(t) = -\frac{F_0}{2\omega} t \cos(\gamma t).$$

In particular, in case (1), if $\gamma \neq \omega$ then

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} (\cos(\gamma t) - \cos(\omega t))$$

is a solution and, in case (2), if $\gamma \neq \omega$ then

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2} (\sin(\gamma t) - \sin(\omega t))$$

is a solution. In both of these, to derive the case $\gamma = \omega$, one can take limits $\gamma \to \omega$ in these expressions.

**Example:** Solve

$$x'' + \omega^2 x = \cos(\gamma t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $\omega = \gamma = 2$ (ie, mechanical resonance). We use SAGE for this:

```sage
sage: t = var('t')
sage: x = function('x', t)
sage: (m,b,k,w,F0) = var("m,b,k,w,F0")
sage: de = lambda y: diff(y,t,t) + w^2 * y - F0 * cos(w * t)
sage: m = 1; b = 0; k = 4; F0 = 1; w = 2
sage: desolve(de(x(t)),[x,t])
'(2*t*sin(2*t)+cos(2*t))/8+%k1 * sin(2*t)+%k2 * cos(2*t)'
sage: desolve_laplace(de(x(t)),["t","x"],[0,0,0])
't*sin(2*t)/4'
sage: soln = lambda t : t*sin(2*t)/4
sage: P = plot(soln(t),0,10)
sage: show(P)
```

This is displayed below:
Figure 1: A forced undamped spring, with resonance.

Example: Solve

\[ x'' + \omega^2 x = \cos(\gamma t), \quad x(0) = 0, \quad x'(0) = 0, \]

where \( \omega = 2 \) and \( \gamma = 3 \) (ie, no mechanical resonance). We use SAGE for this:

```sage
sage: t = var('t')
sage: x = function('x', t)
sage: (m,b,k,w,g,F0) = var("m,b,k,w,g,F0")
sage: de = lambda y: diff(y,t,t) + w^2* y - F0*cos(g*t)
sage: m = 1; b = 0; k = 4; F0 = 1; w = 2; g = 3
sage: desolve_laplace(de(x(t)),["t","x"],[0,0,0])
'cos(2*t)/5-cos(3*t)/5'
sage: soln = lambda t : cos(2 * t)/5-cos(3 * t)/5
sage: P = plot(soln(t),0,10)
sage: show(P)
```

This is displayed below:
Figure 2: A forced undamped spring, no resonance.

**Exercise**: Using SAGE, solve

\[ x'' + \omega^2 x = \cos(t), \quad x(0) = 0, \quad x'(0) = 0, \]

when \( \omega = 1 \) and when \( \omega = 2 \) (or any other value not equal to the resonance frequency).

**References**


