The Laplace transform (LT) of a function $f(t)$, defined on all non-negative numbers $t \geq 0$, is the function $F(s)$, defined by:

$$F(s) = \mathcal{L} \left[ f(t) \right] (s) = \int_0^\infty e^{-st} f(t) \, dt.$$  

This is named for Pierre-Simon Laplace, one of the best French mathematicians in the mid-to-late 18th century [L], [LT]. The LT sends “nice” functions of $t$ (we will be more precise later) to functions of another variable $s$. It has the wonderful property that it transforms constant-coefficient differential equations in $t$ to algebraic questions in $s$.

The LT has two very familiar properties: Just as the integral of a sum is the sum of the integrals, the Laplace transform of a sum is the sum of Laplace transforms:

$$\mathcal{L} \left[ f(t) + g(t) \right] (s) = \mathcal{L} \left[ f(t) \right] (s) + \mathcal{L} \left[ g(t) \right] (s)$$

Just as constant factor can be taken outside of an integral, the LT of a constant times a function is that constant times the LT of that function:

$$\mathcal{L} \left[ af(t) \right] (s) = a \mathcal{L} \left[ f(t) \right] (s)$$

In other words, the LT is linear.

For which functions $f$ is the LT actually defined on? We want the indefinite integral to converge, of course. A function $f(t)$ is of exponential

1These notes licensed under Attribution-ShareAlike Creative Commons license, http://creativecommons.org/about/licenses/meet-the-licenses The graphs were created using SAGE and GIMP http://www.gimp.org/ by the author. Originally written 9-25-2007. Last modified 2008-11-28. Some of the latex code is taken from the excellent (public domain!) text by Sean Mauch [M].

2For all practical purposes, you may assume $f(t) = 0$ for all $t < 0$, if you want a function defined on all of $\mathbb{R}$. 
order \( \alpha \) if there exist a constant \( M \) such that

\[ |f(t)| < Me^{\alpha t}, \quad \text{for all } t \geq 0. \]

Abusing terminology, we say \( f(t) \) is of exponential order if there is a (finite) constant \( \alpha > 0 \) for which \( f \) is of exponential order \( \alpha \). If \( \int_0^t f(t) \, dt \) exists and \( f(t) \) is of exponential order \( \alpha \) then the Laplace transform \( \mathcal{L}[f](s) \) exists for \( s > \alpha \). We shall say that \( f(t) \) is “nice”, in the sense used above, if it is Riemann-integrable and of exponential order. You can see that the image \( F(s) \) of such an \( f(t) \) under the Laplace transform is a function of \( s > \alpha \) which tends to 0 as \( s \to \infty \): for some (possibly huge) constant \( C \), we have \( F(s) \leq C \int_0^\infty e^{at}e^{-st} \, dt = \frac{C}{s-\alpha} \to 0 \) as \( s \to \infty \).

**Example 1** Consider the Laplace transform of \( f(t) = 1 \). The LT integral converges for \( s > 0 \).

\[
\mathcal{L}[f](s) = \int_0^\infty e^{-st} \, dt = \left[-\frac{1}{s}e^{-st}\right]_0^\infty = \frac{1}{s}.
\]

**Example 2** Consider the Laplace transform of \( f(t) = e^{at} \). The LT integral converges for \( s > a \).

\[
\mathcal{L}[f](s) = \int_0^\infty e^{(a-s)t} \, dt = \left[-\frac{1}{s-a}e^{(a-s)t}\right]_0^\infty = \frac{1}{s-a}.
\]

Differentiate both side \( n \) times with respect to \( s \) to get

\[
\mathcal{L}[t^n e^{at}](s) = \int_0^\infty t^n e^{(a-s)t} \, dt = \frac{(-1)^n n!}{(s-a)^{n+1}}. \tag{1}
\]

You can check the first several of these in SAGE as follows:
Define the unit step (Heaviside) function by

\[ u(t - c) = \begin{cases} 
0 & \text{for } t < c \\
1 & \text{for } t > c,
\end{cases} \]

where \( c > 0 \). This models a power source which turns “on” (1) and “off” (0).

**Example 3** Compute the Laplace transform of the unit step function:

\[
\mathcal{L}[u(t-c)](s) = \int_0^\infty e^{-st}u(t-c) \, dt \\
= \int_c^\infty e^{-st} \, dt \\
= \left[ \frac{e^{-st}}{-s} \right]_c^\infty \\
= \frac{e^{-cs}}{s},
\]

for \( s > 0 \).

**Example 4** Consider

\[
f(t) = \begin{cases} 
1, & \text{for } t < 2, \\
0, & \text{on } t \geq 2.
\end{cases}
\]

The plot of this function is displayed below:
Figure 1: The piecewise constant function $1 - u(t - 2)$.

We show how SAGE can be used to compute the LT of this. (Note this function is “on” then turns “off”, whereas the unit step function is “off” then turns “on”).

```
sage: t = var('t')
sage: s = var('s')
sage: f = Piecewise([[(0,2),1],[(2,infinity),0]])
sage: f.laplace(t, s)
1/s - e^(-(2*s))/s
sage: f1 = lambda t: 1
sage: f2 = lambda t: 0
```

According to SAGE,

$$\mathcal{L}[f](s) = 1/s - e^{-2s}/s.$$ 

The inverse Laplace transform is denoted

$$f(t) = \mathcal{L}^{-1}[F(s)](t),$$

where $F(s) = \mathcal{L}[f(t)](s)$. For instance,

$$\mathcal{L}^{-1}[1/(s + 1)](t) = e^{-t}.$$ 

This can be computed in SAGE as follows.

```
sage: t,s = var('t,s')
```

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Next, some properties of the LT.

- Differentiate the definition of the LT with respect to \( s \):

\[
F'(s) = - \int_0^\infty e^{-st} f(t) \, dt.
\]

Repeating this:

\[
\frac{d^n}{ds^n} F(s) = (-1)^n \int_0^\infty e^{-st} t^n f(t) \, dt. \tag{2}
\]

This is called a derivative theorem for the LT.

- In the definition of the LT, replace \( f(t) \) by its derivative \( f'(t) \):

\[
\mathcal{L}[f'(t)](s) = \int_0^\infty e^{-st} f'(t) \, dt.
\]

Now integrate by parts (\( u = e^{-st} \), \( dv = f'(t) \, dt \)):

\[
\int_0^\infty e^{-st} f'(t) \, dt = f(t)e^{-st}|_0^\infty - \int_0^\infty f(t) \cdot (-s) \cdot e^{-st} \, dt = -f(0) + s\mathcal{L}[f(t)](s).
\]

Therefore, if \( F(s) \) is the LT of \( f(t) \) then \( sF(s) - f(0) \) is the LT of \( f'(t) \):

\[
\mathcal{L}[f'(t)](s) = s\mathcal{L}[f(t)](s) - f(0). \tag{3}
\]

- Replace \( f \) by \( f' \) in (3),

\[
\mathcal{L}[f''(t)](s) = s\mathcal{L}[f'(t)](s) - f'(0), \tag{4}
\]

and apply (3) again:

\[
\mathcal{L}[f''(t)](s) = s^2 \mathcal{L}[f(t)](s) - sf(0) - f'(0). \tag{5}
\]
This, and the previous item, is also called a **derivative theorem** for the LT.

- Using (3) and (5), the LT of any constant coefficient ODE

\[ ax''(t) + bx'(t) + cx(t) = f(t) \]

is

\[ a(s^2 \mathcal{L}[x(t)](s) - sx(0) - x'(0)) + b(s \mathcal{L}[x(t)](s) - x(0)) + c \mathcal{L}[x(t)](s) = F(s), \]

where \( F(s) = \mathcal{L}[f(t)](s) \). In particular, the LT of the solution, \( X(s) = \mathcal{L}[x(t)](s) \), satisfies

\[ X(s) = \frac{F(s) + ax(0) + ax'(0) + bx(0)}{as^2 + bs + c}. \]

Note that the denominator is the characteristic polynomial of the DE.

**Moral of the story:** it is always very easy to compute the LT of the solution to any constant coefficient non-homogeneous linear ODE.

**Example 5** Let us solve the DE

\[ x' + x = t^{100}e^{-t}, \quad x(0) = 0. \]

using LTs. Note this would be highly impractical to solve using undetermined coefficients. (You would have 101 undetermined coefficients to solve for!)

First, we compute the LT of the solution to the DE. The LT of the LHS: by (2),

\[ \mathcal{L}[x' + x] = sX(s) + X(s), \]

where \( F(s) = \mathcal{L}[f(t)](s) \). For the LT of the RHS, let

\[ F(s) = \mathcal{L}[e^{-t}] = \frac{1}{s + 1}. \]

By (2),

\[ \frac{d^{100}}{ds^{100}} F(s) = \mathcal{L}[t^{100}e^{-t}] = \frac{d^{100}}{ds^{100}} \frac{1}{s + 1}. \]
The first several derivatives of $\frac{1}{s+1}$ are as follows:

$$\frac{d}{ds} \frac{1}{s+1} = -\frac{1}{(s+1)^2},$$

$$\frac{d^2}{ds^2} \frac{1}{s+1} = 2\frac{1}{(s+1)^3},$$

$$\frac{d^3}{ds^3} \frac{1}{s+1} = -62\frac{1}{(s+1)^4},$$

and so on. Therefore, the LT of the RHS is:

$$\frac{d^{100}}{ds^{100}} \frac{1}{s+1} = 100! \frac{1}{(s+1)^{101}}.$$ 

Consequently,

$$X(s) = 100! \frac{1}{(s+1)^{102}}.$$ 

Using (1), we can compute the ILT of this:

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}\left[100! \frac{1}{(s+1)^{102}}\right] = \frac{1}{101!} \mathcal{L}^{-1}\left[101! \frac{1}{(s+1)^{102}}\right] = \frac{1}{101!} t^{101} e^{-t}.$$ 

**Example 6** Let us solve the DE

$$x'' + 2x' + 2x = e^{-2t}, \quad x(0) = x'(0) = 0,$$

using LTs.

The LT of the LHS: by (3) and (4),

$$\mathcal{L}\left[x'' + 2x' + 2x\right] = (s^2 + 2s + 2)X(s),$$

as in the previous example. The LT of the RHS is:

$$\mathcal{L}\left[e^{-2t}\right] = \frac{1}{s+2}.$$ 

Solving for the LT of the solution algebraically:

$$X(s) = \frac{1}{(s+2)((s+1)^2 + 1)}.$$ 

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The inverse LT of this can be obtained from LT tables after rewriting this using partial fractions:

\[ X(s) = \frac{1}{2} \cdot \frac{1}{s+2} - \frac{1}{2} \cdot \frac{s}{(s+1)^2+1} + \frac{1}{2} \cdot \frac{1}{(s+1)^2+1}. \]

The inverse LT is:

\[ x(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2} \cdot e^{-2t} - \frac{1}{2} \cdot e^{-t} \cos(t) + \frac{1}{2} \cdot e^{-t} \sin(t). \]

We show how SAGE can be used to do some of this.

```sage
t = var('t')
s = var('s')
f = 1/((s+2)*(s+1)^2+1)
f.partial_fraction()
sage: f.partial_fraction()
1/(2*(s + 2)) - s/(2*(s^2 + 2*s + 2))
f.inverse_laplace(s,t)
Inversion: e^(-t)*(sin(t)/2 - cos(t)/2) + e^(-(2*t))/2
```

Exercise: Use SAGE to solve the DE

\[ x'' + 2x' + 5x = e^{-t}, \quad x(0) = x'(0) = 0. \]

References

[L] Wikipedia entry for Laplace:

http://en.wikipedia.org/wiki/Pierre-Simon_Laplace

[LT] Wikipedia entry for Laplace transform:

http://en.wikipedia.org/wiki/Laplace_transform

[M] Sean Mauch, Introduction to methods of Applied Mathematics,

http://www.its.caltech.edu/~sean/book/unabridged.html