SM212 Examination

Time: 180 minutes Fall 1988

1. (a) Show that \( y = x^{-2}\ln(x) \) is a solution for \( x^2y'' + 5xy' + 4y = 0 \).
   (b) Show that \((\cos y + 2x)dx + (3\exp(3y) - x\sin y)dy = 0\) is exact and find its general solution.

2. Consider the first order differential equation \( y' = 1 + x - y; \ y(0) = 1 \).
   
   (a) Draw the direction field for the above equation by first plotting the three isoclines where \( y' = 0, \ y' = -1\), and \( y' = 1 \). Also sketch the solution such that \( y(0) = 1 \) which “fits” your direction field.
   
   (b) Solve the above differential equation and initial condition for its exact solution \( y(x) \) and find \( y(1) \). (Use \( \exp(-1) = .3678794 \).)
   
   (c) Approximate \( y(1) \) by using two steps of Euler’s (constant step) method on the above problem.

3. (a) Solve \( y'' - 6y' + 9y = 2\exp(3x)\); \( y(0) = 1, \ y'(0) = 4 \).
   (b) Give an example of a fourth order linear homogeneous ordinary differential equation with constant coefficients and find its general solution.

4. (a) Use Laplace transforms and partial fractions to solve \( y'' + 4y' + 8y = 8; \ y(0) = 2, \ y'(0) = -2 \).
   (b) Sketch \( f(t) = t + (\cos(t) - t)u(t - \pi) \) and find \( L\{f(t)\} \).
   (c) Use convolution to evaluate \( L^{-1}\{F(s)G(s)\} \) where you choose \( F(s) \) and \( G(s) \).

5. (a) A 4 lb weight is suspended from a spring with spring constant \( k = 7/8 \) lb/ft and damping constant \( \beta = 1/2 \) lb-sec/ft. If the weight is pulled 1 ft below equilibrium and given a velocity of 5 ft/sec upward, find its displacement in the form \( x(t) = \exp(\alpha t)A\sin(\omega t + \phi) \).
   (b) What value of \( \beta \) would give critical damping?
   (d) If \( \beta = 0 \) and there is an external force of \( 10\cos(bt) \), what value of \( b \) would give resonance?
6. Consider the second order equation \( y'' - x(y')^2 - y = 0, \ y(0) = 1, \ y'(0) = 2. \)

(a) Write an equivalent system of two first order differential equations and initial conditions.

(b) Use 2 steps of Euler’s method on the first order system in (i) to approximate \( y(2) \) in the above differential equation.

(b) Write a second order homogeneous Cauchy-Euler differential equation and find its general solution.

7. (a) Write the following system of differential equation in operator notation and solve for \( x(t) \) only \( 2x' + x + y' + y = 0, \ y' + x' - 2x - y = 0. \)

(b) Write a system of 2 differential equations in operator notation involving only \( q_1(t) \) and \( q_2(t) \) for the electric circuit network below. Do not solve!

![Figure 1: A network.](image)

8. (a) Use the Taylor series method to find the first 4 non-zero terms of a series solution to the differential equation \( y'' - (y')^2 = 0; \ y(0) = 1, \ y'(0) = 1. \)

(b) Find the radius of convergence of the Taylor series.

9. (a) Find the Fourier series for the function of period \( 2\pi \) given by \( f(x) = x, \ -\pi < x < \pi. \) Write the series in summation notation and write out its first 4 non-zero terms.

(b) What does the Fourier series converge to if \( x = \pi/2? \)

(c) Use your results from (a) and (b) to find the sum of \( 1 - 1/3 + 1/5 - 1/7 + 1/9 \ldots \).
10. (a) Set up and solve a partial differential equation and conditions satisfied by the temperature \( u(x,t) \) of a thin insulated wire of length 3 and diffusivity constant \( \beta = 1 \), whose left and right ends are kept at 0°, and whose initial temperature is given by \( f(x) = 5 \sin(2\pi x/3) \) throughout the bar. Use separation of variables and show every step clearly.

(b) Find the temperature in the middle of the bar in part (a) 30\( \pi/2 \) seconds later. Leave your answer in terms of exponentials.

Solutions by Mr. Holcomb (written Fall 2008):
1(a). \( y = x^{-2} \ln(x) \), so \( y' = x^{-3} - 2x^{-3} \ln(x) \) and \( y'' = -5x^{-4}6x^{-4} \ln(x) \). \( x^2y'' + xy' + 4y = 5x \left( \frac{1}{x^3} - \frac{2\log(x)}{x^3} \right) + x^2 \left( \frac{6\log(x)}{x^4} - \frac{5}{x^4} \right) + \frac{4\log(x)}{x^2} = 0 \).
1(b). Not on the current syllabus.
2(a). Plot \{1 + x - y = -1\}, \{1 + x - y = 0\}, \{1 + x - y = +1\}, and the direction field. See Sage plot below.

![Figure 2: Direction fields and isocon for y' = 1 + x - y.](image)

(b) \( y = x + e^{-x}, \ y(1) = 1.367... \)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>( h[f(x,y)] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1/2</td>
<td>1</td>
<td>1/4</td>
</tr>
<tr>
<td>1</td>
<td>5/4</td>
<td>3/8</td>
</tr>
</tbody>
</table>

3a) \( y'' - 6y' + 9y = 2e^{3t}, \ y(0) = 1, \ y'(0) = 4 \)
\( s^2Y(s) - s - 4 - 6sY(s) + 6 + 9Y(s) = 2/(s - 3), \)
\( (s^2 - 6s + 9)Y(s) = s - 3 + 1 + 2/(s - 3), \)
\( Y(s) = 1/(s - 3) + 1/(s - 3)^2 + 2/(s - 3)^3 \)
\( y(t) = e^{3t} + t \cdot e^{3t} + t^2 \cdot e^{37} \).

3b) \( y''' = 0. \ y(t) = c_1 + c_2t + c_3t^2 + c_4t^3 \)
4a) \( y'' + 4y' + 8y = 8, \ y(0) = 2, \ y'(0) = -2 \)
\[ s^2Y(s) + 4sY(s) + 8Y(s) = 8/2 + 2s + 6 \]
\[ Y(s) = (s + 2)/((s + 2)^2 + 4) + 1/s \]
\[ y(t) = 1 + e^{-2t} \cos(2t) \]

**4b)** \[ F(s) = 1/s^2 + e^{-\pi s} \mathcal{L}\{\cos(t + \pi) - (t + \pi)\}(s) = 1/s^2 + e^{-\pi s}(-s/(s^2 + 1) - 1/s^2 - \pi/s) \]

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**5a)** \[ m = 4/32 = 1/8, \quad b = 4/8, \quad k = 7/8, \]
\[ mx'' + bx' + kx = 0 \]
\[ x'' + 4x' + 7x = 0, \quad x(0) = 1, \quad x'(0) = -5 \]
roots of char poly: \[-1/2 \pm 3\sqrt{3}/2 \]
\[ x(t) = e^{-2t}(\cos(\sqrt{3}t) - \sqrt{3}\sin(\sqrt{3}t)) \]
\[ x(t) = e^{-2t}A\sin(\sqrt{3}t + \phi) \]
\[ A = \sqrt{1 + 3} = 2 \]
\[ \phi = 2 \tan^{-1}(1/(2 - \sqrt{3})) \]

**5b)** \[ b = \sqrt{7}/4 \]

**5c)** \[ \sqrt{k/m} = \sqrt{7} \]

**6a)**

i) \[ y_1 = y, \quad y_2 = y' \]

\[
\begin{align*}
y_1' &= y_2, & y_1(0) &= 1, \\
y_2' &= y_1 + xy_2, & y_2(0) &= 2.
\end{align*}
\]

ii) \[
\begin{array}{cccc}
t & y_1 & h*f_1(x,y_1,y_2) & y_2 \hline
0 & 1 & 2.0 & 2 \\
1 & 3.0 & 3.0 & 3.0 & 6.0 \\
2 & 6.0 & 9.0 & 9.0 & 24.0 \\
\end{array}
\]

**6b)** Not on the syllabus.

**7a)** Operator notation: \((2D + 1)x + (D + 1)y = 0 \quad (D - 2)x + (D - 1)y = 0, \)**
so \((D + 3)x + 2y = 0\) or \(y = -(D + 3)x/2\)
\((D - 1)(2D + 1)x + (D + 1)(D + 1)y = 0\) \( (D + 1)(D - 2)x + (D + 1)(D - 1)y = 0\)
\((2D^2 - D - 1)x - (D^2 - D - 2)x = 0\), so \((D^2 + 1)x = 0\), so
\(x = c1\cos(t) + c2\sin(t)\) and \(y(t) = (-x'(t) - 3x(t))/2\)

7b)
8a) Not on syllabus
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9a) \(f(x) = x, \ L = \pi\).
\(f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)]\), by symmetry \(a_i = 0\), so \(f(x) \sim \sum_{n=1}^{\infty} b_n \sin(nx)\).
\(b_n = \frac{2}{\pi} \int_0^\pi x \sin(nx) \, dx = -2 \cos(n\pi)/n\).
\(f(x) \sim 2 \sin(x) - \sin(2x) + \frac{2}{3} \sin(3x) - \sin(4x)/2 + -...\)
9b) \(\pi/2\).
9c) \(\pi/4\).
10a) \(f(x) = 5 \sin(2\pi x/3), \ u(x, t) = 5 \sin(2\pi x/3)e^{-2\pi^2 t/9}\),
10b) \(u(3/2, 30\pi/2) = 5 \sin(\pi)e^{-2\pi^2 t/9} = 0\).