Differential Equations
SM212 Spring Semester 2013-2014
Exercises up to Test 3

Free online text:
[D] Paul Dawkins, Differential Equations, http://tutorial.math.lamar.edu/Classes/DE/DE.aspx (also available as downloadable pdf). You can use this as the main text, but it has no exercises.

This text is supplemental.

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The following exercises are mostly taken from [L] or [T] or make up by the instructor.

1. Introduction to Diff. Eqs

Read: The corresponding sections (pages 7-12) in [D] and pages 7-12 in [L].

(a) Find the order of the differential equation: (a) \( y'' + 2y'y''' + x = 0 \),
(b) \( y'' - y' = x^8 \).

(b) Show that \( x = e^{4t} \) is a solution to \( x''' - 12x'' + 48x' - 64x = 0 \).
(c) Show that \( x = e^t \) is not a solution to \( x''' - 12x'' + 48x' - 64x = 0 \).
(d) Determine if the DE is homogeneous or not. (a) \( y'' + 2y' + y = 0 \)
(b) \( y'' + 2y'' + x = 0 \)
(e) Determine if the DE is linear or not. (a) $y'' + 2y' + y = 0$ (b) $y'' + 2y' + y^2 = 0$

(f) Show that $x^2 + y^2 = 1$ is an implicit solution to $dy/dx = -x/y$.

(g) Show that both $y = \sqrt{1 - x^2}$ and $y = -\sqrt{1 - x^2}$ are an explicit solution to $dy/dx = -x/y$.

(h) Show that $\ln(x) = 2 + \sin(t)$ is an implicit solution to $dx/dt = x\cos(t)$.

(i) Verify that the given function is a solution:
   (a) $y'' + 4y = 0, \ y = c_1 \cos(2x) + c_2 \sin(2x)$,
   (b) $y' + 2y = 7, \ y = \frac{7}{2} + ce^{-2x}$,
   (c) $y' = (2 - y)(1 - y), \ ln\left(\frac{2-y}{1-y}\right) = x + c$.

(j) Show that $x = e^t - e^{-t}, \ y = e^t + e^{-t}$ is a solution to the system

\[
\begin{cases}
  x' = y, \\
y' = x.
\end{cases}
\]

(k) Show that $u(x, t) = e^{-t} \sin(x)$ solves $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

2. Initial Value Problems

Read: The corresponding pages 7-12, 71-75 in [D] and pages 7-12 in [L].

(a) From [T], do Problems 1, 3 on pages 60-61: Find all $(x_0y_0)$ for which the existence and uniqueness theorem implies that the IVP $y' = f(x, y), \ y(x_0) = y_0$ has a unique solution on some open interval that contains $x_0$.
   (a) $y' = \frac{x^2 + y^2}{\sin(x)}$,
   (b) $y' = \tan(xy)$.

(b) Show that $x = e^t$ is a solution to $x'' - x = 0. \ x(0) = 1, \ x'(0) = 1$.

(c) Show that $x = e^{-t}$ is not a solution to $x'' - x = 0. \ x(0) = 1, \ x'(0) = 1$.

(d) Show that $y = \cos(x)$ and $y = 1$ both solve $y' = \sqrt{1 - y^2}, \ y(0) = 1$. 

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(e) Consider the IVP \( y' = \sqrt{1 - y^2}, \ y(x_0) = y_0 \). What is the largest set of all \((x_0, y_0)\) under which [D], Theorem 2 on page 72 (the so-called “existence and uniqueness theorem for first-order ODEs”), guarantees that there is a unique solution?

(f) Consider the IVP \( y' = -x/y, \ y(x_0) = y_0 \). What is the largest set of all \((x_0, y_0)\) under which [D], Theorem 2 on page 72 (the so-called “existence and uniqueness theorem for first-order ODEs”), guarantees that there is a unique solution?

(g) Can you find two different explicit solutions to \( dy/dx = -x/y, \ y(1) = 0? \) (Hint: Look at previous exercises.)

(h) Using \( y = c_1 \cos(t) + c_2 \sin(t) \) as the general solution to \( y'' + y = 0 \), find all solutions (if any) to

(a) \( y(0) = 0, \ y(\pi) = 1, \)
(b) \( y(0) = 0, \ y(\pi) = 0, \)
(c) \( y(0) = -1, \ y(\pi/2) = 1. \)

(i) Show that \( x = \frac{e^t - e^{-t}}{2}, \ y = \frac{e^t + e^{-t}}{2} \) is a solution to the system with ICs,

\[
\begin{cases}
x' = y, \ x(0) = 0 \\
y' = x, \ y(0) = 1.
\end{cases}
\]

(j) Show that \( u(x, t) = e^{-t} \sin(x) \) solves

\[
\begin{cases}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \\
u(x, 0) = \sin(x).
\end{cases}
\]

(k) Recall the general solution to the following ODEs were given in a previous exercise. Use them to solve the IVPs below:

(a) \( y'' + 4y = 0, \ y(0) = 0, \ y'(0) = 1, \) sketch the solution for \(-\pi < x < \pi.\)
(b) \( y' + 2y = 7, \ y(0) = 1, \) sketch the solution,
(c) \( y' = (2 - y)(1 - y), \ y(0) = 0, \) sketch the solution,
(d) \( y' = (2 - y)(1 - y), \ y(0) = 1, \) sketch the solution,
(e) \( y' = (2 - y)(1 - y), \ y(0) = 2, \) sketch the solution.
3. Review of separable and 1st order linear

Read pages 20-44 in [D] and pp 22-31 in [L].

(a) From [T] page 41: Do Problems 1, 3: find the general solution.
   (a) $y' + ay = 0$ ($a =$ constant).
   (b) $xy' + (\ln x)y = 0$.

(b) The differential equation $\frac{dy}{dx} = x - 2y^2$ is
   a) separable   b) linear   c) homogeneous   d) none of these.

(c) The differential equation $x^2 \frac{dy}{dx} = e^{x^2+y^2}$ is
   a) separable   b) linear   c) homogeneous   d) none of these.

(d) Find all solutions to $y' = y^2 - 1$, including the constant solutions.

(e) Answer the following true or false.
   (a) The ODE $y' - xy = 2y$ is separable, 1st order, and linear.
       T   F
   (b) The ODE $y'' - xy = 2x$ is separable and linear.
       T   F
   (c) The function $y = 3 \exp(-4x) - \frac{1}{2}$ solves the IVP
       $y' + 4y = 2, \quad y(0) = \frac{5}{2}$.
       T   F
   (d) The ODE $xy' = xy^2 - x, y(1) = -1$ has $y = -1$ as its unique
       solution.
       T   F

(f) Find the general (implicit) solution to $y' = -x/y$.

(g) Find the general (implicit) solution to $v' + kv = mg$, where $k, m, g$
    are positive constants and $v = v(t)$ is the dependent variable.

(h) Find the general (implicit) solution to $Li' + Ri = e$, where $L, R, e$
    are constants and $i = i(t)$ is the dependent variable.

(i) Find the general (implicit) solution to $Rq' + q/C = e$, where $C, R, e$
    are constants and $q = q(t)$ is the dependent variable.
(j) Find the general (implicit) solution to $T' = k(T - T_{room})$, where $k, T_{room}$ are constants and $T = T(t)$ is the dependent variable.

4. Applications (falling body/cooling/circuits/mixing)

Read pages 76-88 in [D].

(a) From [T] page 148, do Problems 1-4:

(a) A thermometer is moved from a room where the temperature is 70F to a freezer where the temperature is 12F. After 30 seconds the thermometer reads 40F. What does it read after 2 minutes?

(b) A fluid initially at 100C is placed outside on a day when the temperature is −10C, and the temperature of the fluid drops 20C in one minute. Find the temperature $T(t)$ of the fluid for $t > 0$.

(c) At 12 : 00 PM a thermometer reading 10F is placed in a room where the temperature is 70F. It reads 56 when it’s placed outside, where the temperature is 5F, at 12 : 03. What does it read at 12 : 05 PM?

(d) A thermometer initially reading 212F is placed in a room where the temperature is 70F. After 2 minutes the thermometer reads 125F.

(i) What does the thermometer read after 4 minutes?

(ii) When will the thermometer read 72F?

(iii) When will the thermometer read 69F?

(b) A parachutist weights 250 lbs (with chute). The chute is released at the time which the velocity is 50 ft/sec and the distance from the ground is 1000 ft. The force due to air resistance is proportional to velocity, $\vec{F}_{res} = -4\vec{v}$. Find the time of impact.

(c) Sherlock Holmes is awoken by a phone call from a policeman at 3am. A body has been discovered and foul play is suspected. Sherlock tells the police to determine the temperature of the body and, when he arrives at the scene 30 minutes later, he takes the temperature again. The two readings on that cold 60 degree F morning were 90 degrees F and 80 degrees F. Find the time of death (use 98.6 degrees F for the normal body temperature).
(d) A circuit has a 1 henry inductor in series with a 4 ohm resistor. A battery of 2 volts is attached. Assume that the initial current is $5/2$ amps. Find (a) the differential equation governing the current, (b) the current at time $t$.


(f) Wil. E. Coyote weights 128 lbs (with chute). The chute is released 15 seconds after he jumps from a height of 1000 ft. The force due to air resistance with chute closed is proportional to velocity and is given by $F_{res} = -8v$.

Find the time of impact within 1 second of accuracy.

(g) Curly, Moe, and Larry are incompetent butlers for Daffy Duck, who orders then to prepare his bath. The bathtub holds 100 gallons when filled. Curly begins pouring in a soap water solution at a rate of 1/2 gallon/minute. The soap concentration is 1/4 lb/gallon in his soap water solution. Larry, in charge of putting the plug in the drain, screwed up and now the bath water is pouring down the drain at a rate of 1 gallon/minute. Originally, Moe filled the bath full of pure water. Find the amount (in lbs) of soap in the bathtub after 100 minutes.

5. Numerical Solutions

Read pages 8-18 and 93-100 in [D].

(a) From [T] page 107, do Problems 1, 3: use Euler’s method to find approximate values of the solution of the given initial value problem at the points $x_i = x_0 + ih$, where $x_0$ is the point where the initial condition is imposed and $i = 1, 2, 3$. The purpose of these exercises is to familiarize you with the computational procedure of Euler’s method.

(a) $y' = 2x^2 + 3y^2 - 2; \ y(2) = 1, \ h = 1/20$, 
(b) $y' + 3y = x^2 - 3xy + y^2, \ y(0) = 2, \ h = 1/20$.

(b) Suppose $y' = xy, \ y(0) = 1$.

(a) Using isoclines and direction fields, graph an approximation to the solution on $0 < x < 1$.

(b) Solve for $y(x)$ explicitly and sketch its graph over $0 < x < 1$.  

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(c) Compute \( y(1) \) with at least 2 decimal places of accuracy.

(c) (no calculator) Using \( h = 1/2 \) in Euler’s method, approximate \( y(1) \), where \( y’ = y^2 - 3y + 2, \ y(0) = 0 \).

(d) Construct a direction field and plot some integral curves in the indicated rectangular region.

(a) \( y’ = y(y - 1), \ (x, y) | \ 0 \leq x \leq 2, \ -2 \leq y \leq 2 \)

(b) \( y’ = y - 1, \ (x, y) | \ 0 \leq x \leq 2, \ -2 \leq y \leq 2 \)

(c) \( y’ = -(y - 1), \ (x, y) | \ 0 \leq x \leq 2, \ -2 \leq y \leq 2 \)

(d) \( y’ = x - 1, \ (x, y) | \ 0 \leq x \leq 2, \ -2 \leq y \leq 2 \)

6. Homogeneous w/const. coeffs

(a) From [T] pages 203-204, do Problems 1, 3, 5, 10.


(c) From [T] page 277-278, do Problems 1,3,4,5. (Spring-mass systems)

(d) From [T] page 288, do Problems 1,2,7,9. (Spring-mass systems)

(e) From [T] page 295, do Problems 1-4. (LRC circuits)

7. Test 1

8. Undetermined Coefficients or Annihilators


(b)  
  ii. From [T] page 278: Problems 10,15. (Spring-mass systems)
  iii. From [T] page 288: Problems 13, 20. (Spring-mass systems)

9. Laplace Transform

Read Dawkins [d], pp. 188-208.

(a) From [T] page 403: Problems 1- 5.
(b) First “shifting” or “translation” theorem: \( \mathcal{L}[e^{at}f(t)](s) = \mathcal{L}[f(t)](s-a) \).

Exercise: Using only the definition of the LT (and not the math tables), compute \( \mathcal{L}[\cos(5t)](s) \) (hint: compute the integral using integration by parts twice).

Exercise: Use the first translation theorem to compute \( \mathcal{L}[e^{-2t}\cos(5t)](s) \).

10. Inverse Laplace/Derivatives

(a) From [T] page 412-413: Problems 1-5.

(b) First derivative theorem: \( \mathcal{L}[f'(t)](s) = s\mathcal{L}[f(t)](s) - f(0) \). Use the first derivative theorem to compute \( \mathcal{L}[\exp(-3t)](s) \). In other words, plug \( f(t) = \exp(-3t) = e^{-3t} \) into the first derivative theorem and solve for \( F(s) \).

(c) Derivative theorem applied twice: \( \mathcal{L}[f''(t)](s) = s^2\mathcal{L}[f(t)](s) - f'(0) - f(0)s \). Use the derivative theorem to compute \( \mathcal{L}[\sin(5t)](s) \). In other words, plug \( f(t) = \sin(5t) \) into both sides of the derivative theorem (use the fact that \( \sin(5t)'' = -25 \sin(5t) \)) and solve for \( F(s) \).

(d) Suppose \( x = x(t) \) satisfies \( x' + x = 2, x(0) = -1 \). Take LTs and use the first derivative theorem to solve for \( \mathcal{L}[x(t)](s) \).

(e) From [T] page 419: Problems 1, 3, 5, 7.

11. Test 2

12. Unit step functions
   Read Dawkins pp. 209-221.

(a) From [T] page 428: Problems 1-4. (LTs of piecewise-defined functions.)

(b) From [T] page 438: Problems 1-5. (Using LTs to solve constant-coefficient 2nd order ODEs, with piecewise-defined forcing function.)

13. Delta functions, Convolution thrm, impulse-response
   Read Dawkins pp. 240-247.
(a) From [T] page 461: Problems 1-5. (Dirac delta)
(b) From [T] page 449-450: Problems 1 (a)-(d), 2 (a)-(d), 3 (a)-(d).
   (convolution)
(c) Give an example of [T], page 450, Problem 8.

14. Introduction to matrices

   Dawkins: pages 242-257
   Lebl [L], §3.2 pp 90-97

   L page 97: Problems 3.2.2-3.2.10.
   (a) matrix arithmetic
      [L] page 97: Problems 3.2.2-3.2.10.
   (b) row reduction, inverses, determinants
      [L] page 97: Problems 3.2.2-3.2.10.

15. Solving linear systems via Laplace transforms

   Problems: Solve all problems using LTs.

   (a) From [T] page 519: Problem 1, 3, 5. (Introduction to linear systems.)
   (b) From [T] page 549-550: Problems 1, 2, 16, 17
   (c) From [T] page 567-568: Problems 1, 3, 17, 18

16. Application: Lanchester’s model and electrical networks

   (a) A battle is modeled by

   \[
   \begin{cases}
   x' = -4y, & x(0) = 150, \\
   y' = -x, & y(0) = 90.
   \end{cases}
   \]

   (a) Write the solutions in parameteric form. (b) Who wins? When? State the losses for each side.
   (b) Problems 15, 16 in section 7.6 (page 299) in Zill+Cullen.

17. Test 3
18. Solving linear systems via eigenvalues and eigenvectors
   Dawkins: pages 258-267
   Problems: Solve all problems using the eigenvalue method.
   (a) From [T] page 549-550: Problems 1, 2, 16, 17 (real distinct eigenvalues)
   (b) From [T] page 567-568: Problems 1, 3, 17, 18 (complex eigenvalues)

19. Euler’s Method for systems
   Problems 1, 2, 6 in section 9.4 (page 357) in Zill+Cullen. Use Euler’s method for each of them (not Runga-Kutta).

20. Fourier Series
   Orthogonality: Dawkins, pp 397-404
   Fourier Series: Dawkins, pp 425-438
   (a) Let
      \[ f(x) = \begin{cases} 
      1, & 0 \leq t \leq \pi, \\
      -1, & -\pi < t < 0, 
      \end{cases} \]
      period 2\( \pi \).
      • Find the Fourier series of \( f(x) \) and graph it, \(-2\pi < x < 2\pi\).
      • Write down the series in summation notation and write down the sum of the first 3 non-zero terms.
      • Sketch the function, sketch the sum of the first 3 non-zero terms.
   (b) Let
      \[ f(x) = \begin{cases} 
      1, & -\pi < x < \pi/2, \\
      0, & \pi/2 < x < \pi 
      \end{cases} \]
      period 2\( \pi \).
      • Graph the Fourier series of \( f(x) \) over \(-2\pi < x < 2\pi\).
      • Compute the Fourier series of \( f(x) \).
      • Write your answer in summation notation and write down the sum of the first 3 non-zero terms.
• Sketch $f(x)$ and the sum of the first 3 non-zero terms.

21. Fourier Sine and Cosine Series

Sine Series and Cosine Series: Dawkins, pp 405-424

(a) Let

$$f(x) = \begin{cases} 
1, & 0 \leq t \leq \pi/2, \\
-1, & \pi/2 < t < \pi.
\end{cases}$$

• Find the Sine series of $f(x)$ and graph it, $-2\pi < x < 2\pi$.
• Write down the series in summation notation and write down the sum of the first 3 non-zero terms.
• Sketch the function, sketch the sum of the first 3 non-zero terms.

(b) Let

$$f(x) = \begin{cases} 
1, & 0 \leq t \leq \pi/2, \\
-1, & \pi/2 < t < \pi.
\end{cases}$$

• Find the Cosine series of $f(x)$ and graph it, $-2\pi < x < 2\pi$.
• Write down the series in summation notation and write down the sum of the first 3 non-zero terms.
• Sketch the function, sketch the sum of the first 3 non-zero terms.

22. Separation of variables for PDEs

Dawkins: pp 453-463

23. Heat equation, zero ends

Dawkins: pp 464-476

24. Heat equation, insulated ends

Dawkins: pp 464-476

25. Test 4