1. Classification
   - Order
   - Linearity: \(a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = f(x)\), where dependent variable \(y\) and its derivatives have no nonlinear operations (e.g., squaring) performed on them.
   - Separable
   - Autonomous

2. Graphical approximations to the solution to a first order ODE
   - Direction fields and isoclines
   - Autonomous DEs
     - \(y' = f(y)\),
     - Find critical points and sketch phase portrait
     - Types of equilibria
       (a) stable - attractor
       (b) unstable - repellor

3. Numerical methods for \(y' = f(x, y)\)
   - Euler’s method
     - \(y_{new} = y_{old} + hf(x_{old}, y_{old}), \quad x_{new} = x_{old} + h\)
     - or \(y_{n+1} = y_n + hf(x_n, y_n), \quad x_{n+1} = x_n + h\)
   - successive “tangent line”/linear approximation using slope = \(y' = f(x, y)\)
   - Improved Euler’s method
     - If \(y_{new}^* = y_{old} + hf(x_{old}, y_{old})\) then
       \(y_{new} = y_{old} + \frac{h}{2}[f(x_{old}, y_{old}) + f(x_{new}, y_{new}^*)], \quad x_{new} = x_{old} + h\)
– successive averaged “tangent line”/linear approximation using slope \( y' = f(x, y) \)

4. First Order Methods of Solution

- Separation of Variables \( y' = f(x)/g(y) \) \( \implies \int g(y) \, dy = \int f(x) \, dx \)

- Integrating Factor
  
  - \( y' + p(x)y = q(x) \)
  
  - \( \mu = e^{\int p(x) \, dx} \)
  
  - \( \mu y' + p(x)\mu y = (\mu y)' = \mu f(x) \)
  
  - \( y = \frac{\int \mu f(x) \, dx + C}{\mu} \)

5. Applications

- Exponential Growth/Decay
  
  - General \( y' = Ay, \ y(0) = y_0 \) \( \implies \ y = y_0 e^{At} \)
  
  - Radioactive Decay \( m(t) = m_0 (\frac{1}{2})^{t/\tau} \)

- Heating/Cooling
  
  \( T' = k \cdot (T - T_{room}) \), where \( T = T(t) \) is the temperature of the object and \( T_{room} \) (which can depend on \( t \)) is temperature of the room (or environment or medium)

- Mixing \( A' = F_{in}C_{in} - F_{out} \frac{A}{\text{Tank}(t)} \), where \( F_{in/out} \) is the flow rate of the solution flowing in/out, \( C_{in} \) is the concentration of the solution pouring in, \( \text{Tank}(t) \) is the volume of solution in the tank and time \( t \), and \( A = A(t) \) is the amount (mass) of solute.

- Falling Body \( mv' + kv = mg \), where \( k \geq 0 \) is the coefficient of air resistance.

- Circuits
  
  - RL \( L \frac{di}{dt} + Ri = e(t) \), solve for \( i = i(t) \)
  
  - RC \( R \frac{dq}{dt} + QC = e(t) \), solve for \( q = q(t) \)

6. Higher Order DE Underlying Theory
- $n^{th}$ Order Differential Equation
  Initial Value Problem, subject to $n$ initial conditions

  \[ a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = f(x), \]

  Initial Conditions: $y(x_0) = y_0$, $y'(x_0) = y_1$, $\ldots$, $y^{(n-1)}(x_0) = y_{n-1}$.

- $n^{th}$ Order Homogeneous Differential Equation

  \[ a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0. \]

  - There are exactly $n$ fundamental solutions, $y_1, y_2, \ldots y_n$
  - Fundamental solutions are linearly independent
  - General solution: $y = c_1 y_1 + \ldots c_n y_n$, for arbitrary constants
    (sometimes called “parameters”) $c_1, \ldots, c_n$

- $n^{th}$ order non-homogeneous differential equation

  \[ a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = f(x), \]

  General solution: $y = y_h + y_p$, where

  - $y_h$ is a solution to
    
    \[ a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = 0, \]

  - $y_p$ is any solution to
    
    \[ a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = f(x). \]

7. Methods of Solution for Second Order Linear Differential Equations

- Homogeneous with Constant Coefficients

  - Factor characteristic polynomial (sometimes called the auxiliary equation)
  - Roots, $r_i$, yield members of Fundamental Set $y_k = e^{rx}$
  - For roots repeated $k$ times, $y_1 = e^{rx}$, $y_2 = xe^{rx}$, $y_3 = x^2 e^{rx}$, $\ldots$, $y_k = x^{k-1} e^{rx}$,
For complex conjugate roots, \( r = \alpha + i\beta \), \( y_1 = e^{\alpha x} \cos(\beta x) \), \( y_2 = e^{\alpha x} \sin(\beta x) \).

- Non-homogeneous with Constant Coefficients (undetermined coefficients and annihilators)

These methods only applies to the non-homogeneous constant coefficient ODEs

\[
a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(x)
\]

where the “forcing function” \( f(x) \) is “elementary”. More precisely, \( f(x) \) must be a sum of terms which are a product of polynomials, exponentials, sin’s and/or cos’s.

**Undetermined coefficients:**

- First find \( y_h \), the solution to the homogeneous equation
- Next find the repeated derivatives of the forcing \( f(x) \), writing down all the individual terms separately, removing constant factors which might be multiplying such terms. By the hypothesis on \( f(x) \), only a finite number of such functions can arise.
- “Guess” for \( y_p \), a linear combination of such functions. The coefficients in this linear combination are the “undetermined coefficients”. Multiply by \( x \) those terms which “agree” with any terms in \( y_h \).
- Plug \( y_p \) into the ODE and solve for the undetermined coefficients
- \( y = y_h + y_p \)
- Solve for the “parameters” \( c_1, \ldots, c_n \) if you are given ICs.

**Annihilator method:** Write

\[
a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(x)
\]

symbolically as \( L(y) = f(x) \).

- Find the Annihilator of \( f(x) \) - the differential operator \( L_0 \) of smallest degree such that \( L_0(f(x)) = 0 \).
- Multiply annihilators for sums of functions
- Find \( y_h \) - the solution to \( L(y) = 0 \).
– Find solutions of \( L_0(L(y)) = 0 \).
  * identify terms which comprise \( y_h \)
  * remaining terms comprise \( y_p \)
  * \( y = y_h + y_p \)
– Use \( L(y) = f(x) \) to solve for coefficients in \( y_p \)
– Solve for “parameters” of \( y_h \) using initial conditions, if given.

8. Applications

- **Free Undamped Motion**
  \( mx'' + kx = 0 \)
  if \( \omega = \sqrt{k/m} \) then
  \[
  x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A \sin(\omega t + \phi)
  \]
  - \( \omega = \) angular speed
  - \( P = 2\pi/\omega = \) period
  - \( f = 1/P = \) frequency
  - \( \phi = 2 \tan^{-1}(\frac{c_2}{c_1}) = \) phase angle
  - \( A = \sqrt{c_1^2 + c_2^2} = \) amplitude

- **Free Damped Motion**
  \( mx'' + bx' + kx = 0 \)
  Roots \( r_1, r_2 \) of \( mD^2 + bD + k = 0 \):
  - real, distinct \( \implies \) over-damped, \( x = c_1 e^{r_1 t} + c_2 e^{r_2 t} \),
  - repeated root \( r_1 = r_2 \) \( \implies \) critically damped, \( x = c_1 e^{rt} + c_2 t e^{rt} \),
  - complex conjugate roots \( r_1 = \alpha + i\beta, \ r_2 = \alpha - i\beta \) \( \implies \) under-damped \( x = c_1 e^{\alpha t} \cos(\omega t) + c_2 e^{\alpha t} \sin(\omega t) \)

- **Forced Motion**
  \( mx'' + bx' + kx = f(t), \) where \( f(t) \) is the external force acting on the spring-mass system.
  Roots \( r_1, r_2 \) of \( mD^2 + bD + k = 0 \)
  - Solve as other non-homogeneous equations: write solution as \( x = x_h + x_p \), where
* $x_h$ is transient term
* $x_p$ is steady state term

Undamped forced motion will resonate if “forced natural frequency” equals “natural frequency”. In other words, if $f(t) = f_0 \sin(\gamma t)$ or $f(t) = f_0 \cos(\gamma t)$, for some $\gamma$, the resonance occurs if and only if $\gamma = \sqrt{k/m}$ (assuming $b = 0$).

- RLC Electric Circuit

$$Lq'' + Rq' + \frac{1}{C}q = e(t)$$

where $e(t)$ is the battery or EMF.
- Solve using same method as Forced Damped Motion
- Don’t forget that $i = q'$
- If you write solution as $q = q_h + q_p$, where
  * $q_h$ is the transient charge
  * $q_p$ is the steady state charge

9. Laplace transforms

- The Laplace Transform converts a constant-coefficient linear differential equation in the independent variable $t$ to an algebraic equation in independent variable $s$.
- The Laplace Transform is an integral transform.

Know the definition and be familiar with the Laplace Transforms of the common functions (polynomials, sines, cosines, exponentials).

Properties of the Laplace Transform:
- Translation theorem 1: $\mathcal{L}[e^{at}f(t)](s) = F(s - a)$
- Translation theorem 2: $\mathcal{L}[f(t-a)u(t-a)](s) = e^{-as}F(s)$
  Think of $u(t-a)$ as a mathematical switch, which turns ON at $t = a$
- Derivative theorem 1: $\mathcal{L}[f'(t)](s) = sF(s) - f(0)$, and similar formulas for $f''(t)$, $f'''(t)$, ...
- Derivative theorem 2: $\mathcal{L}[tf(t)](s) = -F'(s)$, and similar formulas for $t^2 f(t)$, $t^3 f(t)$, ...
DO NOT get confused and take the products of the Laplace Transforms!

Use Laplace transforms to solve initial value problems

* Take Laplace Transform of entire problem
* Solve for Laplace Transform of dependent variable
* Take Inverse Laplace Transform for solution to IVP

Convolution theorem (the Laplace transform of the convolution is the product of the Laplace transforms)

* Know the definition of the convolution
* Can use the convolution theorem to solve second order linear ODEs with constant coefficients

\[ ay' + by' + cy = f(t), \quad y(0) = y'(0) = 0, \]

\[ y(t) = (h * f)(t), \]

where \( f(t) \) is the forcing function and

\[ h(t) = \mathcal{L}^{-1}\left[ \frac{1}{as^2 + bs + c} \right](t) \]

is called the impulse response function.

10. Matrix operations

• Basics

− Augmented Matrix represents a system of equations. For example, the 2 × 2 linear system in standard form (with all the unknowns on the left hand side and the remaining known quantities on the right hand side)

\[
\begin{cases} 
ax + by = r_1 \\
rx + dy = r_2 
\end{cases}
\]

can be represented as its “2 × 3 matrix of coefficients”

\[
A = \begin{pmatrix} 
a & b & r_1 \\
c & d & r_2 
\end{pmatrix}
\]
– In row reduced echelon form (rref), the above is reduced to a much “sparser” matrix, rref(A), which represents the matrix of coefficients of a much simpler linear system.

– Eigenvalues and eigenvectors. The eigenvalue equation forms the definition: \( A\vec{v} = \lambda \vec{v} \). This means \( \vec{v} \) is an eigenvector of \( A \) with eigenvalue \( \lambda \). An \( n \times n \) matrix has \( n \) eigenvalues, counted according to multiplicity: they are the roots of the characteristic polynomial \( p(\lambda) = \det(A - \lambda I) \).

• Application of rref to systems of linear ordinary differential equations with constant coefficients.

  – Take the Laplace transform of all the equations and put it in standard form.
  – Compute the row reduced echelon form of its augmented matrix.
  – Solve for the Laplace transforms of the dependent variables.
  – Take inverse Laplace transforms to solve the system of ODEs.

• Application of eigenvalues and eigenvectors to systems of linear ordinary differential equations with constant coefficients.

  – Put the systems in matrix form: \( \dot{\vec{X}} = A\vec{X} \), where \( \vec{X} = \vec{X}(t) \) is the vector of \( n \) unknown functions (the dependent variables of the system) and \( A \) is an \( n \times n \) matrix of constants.
  – Compute the eigenvalues \( \lambda_1, ..., \lambda_n \), and their corresponding eigenvectors \( \vec{v}_1, ..., \vec{v}_n \).
  – If all the eigenvalues are distinct, the solution is

\[
\vec{X} = c_1 \vec{v}_1 e^{\lambda_1 t} + ... + c_n \vec{v}_n e^{\lambda_n t},
\]

for arbitrary constants (or “parameters”) \( c_1, ..., c_n \).
  – If there are initial conditions, solve for the \( c_1, ..., c_n \).

• Applications

  – Electrical Networks.
    Determine System of independent ODEs using Kirchoffs Laws for current Loops and nodes.
  – Lanchester’s equations
If the X-men are battling the Y-men in a simple conventional battle then
\[
\begin{cases}
  x' = -Ay \\
  y' = -Bx
\end{cases}
\]
can be used to model the number \( x = x(t) \) of X-men at time \( t \) and the number \( y = y(t) \) of Y-men at time \( t \).

- Numerical methods
  - Euler's method for systems
    \[
    \begin{align*}
    y_1' &= f_1(x, y_1, y_2), \quad y_1(a) = c_1, \\
    y_2' &= f_2(x, y_1, y_2), \quad y_2(a) = c_2.
    \end{align*}
    \]
    Use
    \[
    \begin{align*}
    y_{1,\text{new}} &= y_{1,\text{old}} + hf_1(x_{1,\text{old}}, y_{1,\text{old}}, y_{2,\text{old}}), \\
    y_{2,\text{new}} &= y_{2,\text{old}} + hf_2(x_{1,\text{old}}, y_{1,\text{old}}, y_{2,\text{old}}),
    \end{align*}
    \]
    and \( x_{\text{new}} = x_{\text{old}} + h \).

  - For a 2nd order ODE,
    \[
    y'' + p(x)y' + q(x)y = f(x), \quad y(a) = c_1, \quad y'(a) = c_2,
    \]
    do the following:
    * write 2nd Order ODE as a system of two 1st order DEs.
      Using \( y_1 = y \) and \( y_2 = y' \),
      \[
      \begin{align*}
      y_1' &= y_2, \quad y_1(a) = c_1, \\
      y_2' &= f(x) - q(x)y_1 - p(x)y_2, \quad y_2(a) = c_2.
      \end{align*}
      \]
    * Apply Euler’s method for systems as above.

11. Fourier Series

   Represents “any” function on an interval \((-L, L)\) centered at the origin as a convergent series of orthogonal functions.

   \[
   f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos \left( \frac{n\pi x}{L} \right) + b_n \sin \left( \frac{n\pi x}{L} \right)],
   \]
where
\[ a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx, \]
\[ b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx. \]

12. Half-Range Fourier Series

Represents a function defined on an interval \((0, L)\).

- Fourier Cosine Series
  Also used as Fourier series for EVEN functions.
To have a cosine series you must be given two things: (1) a “period” \( P = 2L \), (2) a function \( f(x) \) defined on the interval of length \( L \), \( 0 < x < L \).

\[ f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right), \]

where
\[ a_n = \frac{2}{L} \int_{0}^{L} \cos\left(\frac{n\pi x}{L}\right) f(x) \, dx. \]

- Fourier Sine Series
  Also used as Fourier series for ODD functions.
To have a sine series you must be given two things: (1) a “period” \( P = 2L \), (2) a function \( f(x) \) defined on the interval of length \( L \), \( 0 < x < L \).

\[ f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \]

where
\[ b_n = \frac{2}{L} \int_{0}^{L} \sin\left(\frac{n\pi x}{L}\right) f(x) \, dx. \]
13. Separation of variables for 1st and 2nd order linear homogeneous PDEs

\[ a_1 u_{xx} + a_2 u_{xy} + a_3 u_{yy} + a_4 U_x + a_5 U_y + a_6 u = 0, \quad u = u(x, y), \]

where the coefficients \( a_i \) could depend on \( x \) or \( y \).

Typical examples:

- advection equation
  \[ u_x + cu_t = 0 \]

- heat equation
  \[ ku_{xx} = u_t \]

- wave equation
  \[ \alpha^2 u_{xx} = u_{tt} \]

14. Heat equation

\[
\begin{align*}
\left\{ \begin{array}{l}
k \frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial u(x, t)}{\partial t} \\
 u(x, 0) = f(x), \\
 u(0, t) = u(L, t) = 0.
\end{array} \right.
\]

Here \( u(x, t) \) denotes the temperature at a point \( x \) on the wire at time \( t \), so \( f(x) \) is the wire’s initial temperature.

- zero ends

\[
\begin{align*}
\left\{ \begin{array}{l}
k \frac{\partial^2 u(x, t)}{\partial x^2} = \frac{\partial u(x, t)}{\partial t} \\
 u(x, 0) = f(x), \\
 0 \leq x \leq L, 0 \leq t \leq T.
\end{array} \right.
\]

- Find the sine series of \( f(x) \):

\[ f(x) \sim \sum_{n=1}^{\infty} b_n(f) \sin\left(\frac{n\pi x}{L}\right), \]

- The solution is

\[ u(x, t) = \sum_{n=1}^{\infty} b_n(f) \sin\left(\frac{n\pi x}{L}\right) \exp\left(-k\left(\frac{n\pi}{L}\right)^2 t\right). \]
• insulated ends

\[
\begin{cases}
  k \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t} \\
  u(x,0) = f(x), \\
  u_z(0,t) = u_x(L,t) = 0.
\end{cases}
\]

• Find the cosine series of \( f(x) \):

\[
f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n(f) \cos\left(\frac{n\pi x}{L}\right),
\]

• The solution is

\[
u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n(f) \cos\left(\frac{n\pi x}{L}\right) \exp(-k\frac{n\pi}{L}^2 t).
\]

15. Wave equation

The wave equation with zero ends boundary conditions models the motion of a (perfectly elastic) guitar string of length \( L \):

\[
\begin{cases}
  \frac{\partial^2 w(x,t)}{\partial x^2} = a^2 \frac{\partial^2 w(x,t)}{\partial t^2} \\
  w(0,t) = w(L,t) = 0.
\end{cases}
\]

Here \( w(x,t) \) denotes the displacement from rest of a point \( x \) on the string at time \( t \). The initial displacement \( f(x) \) and initial velocity \( g(x) \) at specified by the equations

\[
w(x,0) = f(x), \quad w_t(x,0) = g(x).
\]

• Find the sine series of \( f(x) \) and \( g(x) \):

\[
f(x) \sim \sum_{n=1}^{\infty} b_n(f) \sin\left(\frac{n\pi x}{L}\right), \quad g(x) \sim \sum_{n=1}^{\infty} b_n(g) \sin\left(\frac{n\pi x}{L}\right).
\]

• The solution is

\[
w(x,t) = \sum_{n=1}^{\infty} \left( b_n(f) \cos\left(\frac{n\pi t}{aL}\right) + \frac{aLb_n(g)}{n\pi} \sin\left(\frac{n\pi t}{aL}\right) \sin\left(\frac{n\pi x}{L}\right) \right).
\]