Fourier Transforms and Convolutions
By: 1/C Nick Culver
Who Was Fourier??

- Noticeably gifted by age 14
- Priest or Mathematician?
- Math it is
- Taught at the Collège de France
- Joined Napoleon’s Army
- OIC of discoveries in Egypt
- Poisson and Biot
What is Fourier Analysis?

- Fourier analysis allows a system to be separated or decomposed into components made up of simpler inputs.
- For example, a function $f(t)$ is a function in time but via a Fourier transform becomes $f(w)$, where omega is a frequency.
Convolution:

- Steward describes convolution as “the distribution of one function in accordance with the law specified by another function (85).”
Convolution cont.

- Overlaps that are a result of spreading and smearing of a function
Definition of convolution and theorem

If $f$ and $g$ are two functions with $\|f\|$ and $\|g\|$ finite, the convolution of $f$ and $g$ is denoted by $f * g$ and is defined by:

$$f * g(x) = \int_{-\infty}^{\infty} f(s) g(x-s) \, ds$$

PROOF (The Convolution Theorem and its Applications)

$$f * g(x) = \int_{-\infty}^{\infty} f(s) g(x-s) \, ds$$

Let $w = u - x$

$$f * g(x) = \int_{-\infty}^{\infty} f(s) g(x-s) \, ds e^{2\pi i su} du$$

$$= \int_{-\infty}^{\infty} f(s) g(u-x) \, ds e^{2\pi i su} du$$

$$= \int_{-\infty}^{\infty} f(x-w) g(w) e^{2\pi i wu} \, dw$$
Convolution Theorem. The Fourier Transform of the convolution of two functions \( f \) and \( g \) is the multiplication of the Fourier transform of \( f \) with the Fourier transform of \( g \):

\[
\hat{f} \ast \hat{g} = \hat{f} \cdot \hat{g}
\]

Convolution Theorem - Inverse: The inverse Fourier transform of \( \hat{f} \cdot \hat{g} \) is the convolution of \( f \ast g \)

**Proof**

Using the inverse Fourier Transform.

\[
\mathcal{F}^{-1}\left\{ \hat{f} \cdot \hat{g} \right\} = \mathcal{F}^{-1}\left\{ \hat{f} \right\} \cdot \mathcal{F}^{-1}\left\{ \hat{g} \right\}
\]

Now interchange the integrals and add the exponents. pg 177

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(s) \cdot g(u+s) e^{-2\pi i (s+u)x} \, ds \, du
\]

This is the integral of \( \hat{g} \cdot \hat{f} \) with the variable \( s \). So we get \( g \ast f \) for that integral.

Thus, \( \mathcal{F}^{-1}\left\{ \hat{f} \cdot \hat{g} \right\} = f \ast g \)

Change the sum exponentials to a product of exponentials

\[
= \int_{-\infty}^{\infty} f(x-w) e^{2\pi i x w} \cdot g(w) e^{2\pi i w x} \, dx \, dw
\]

\[
= \int_{-\infty}^{\infty} f(x-w) e^{2\pi i x w} \, dx \cdot \int_{-\infty}^{\infty} g(w) e^{2\pi i w x} \, dw
\]

\( w \) is a dummy variable so replace \( w \) with \( x \)

\[
\ast \mathcal{F}^{-1}\left\{ \hat{f} \cdot \hat{g} \right\} = \int_{-\infty}^{\infty} f(x-w) e^{2\pi i x w} \cdot g(w) e^{2\pi i w x} \, dw
\]

Thus \( \ast \mathcal{F}^{-1}\left\{ \hat{f} \cdot \hat{g} \right\} = f \ast g \cdot g \ast f \)
Parseval's Equality

**Energy Conservation Statement**

**Theorem.** (Walker 102)

We define $\sum_{n=1}^{\infty} a_n \sin\frac{nx}{L}$ as the Fourier sine series for the function $g$.

where $\|g\|^2 < \infty$ if and only if $\sum_{n=1}^{\infty} |a_n|^2$ is convergent. Thus introducing

*Parseval's equality:*

$$\frac{1}{L} \sum_{n=1}^{L} |a_n|^2 = \int_0^L |g(x)|^2 \, dx = \|g\|^2$$

In physics, the relation is normally written as.

Given $f^x = \frac{1}{\sqrt{2\pi}} \int f(x) e^{ikx} \, dx$ and $\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-ikx} \, dx$. (Prof. Tankersley)

$$g^x = \frac{1}{\sqrt{2\pi}} \int g(x) e^{imx} \, dx$$

Then Parseval's equality becomes.

$$\overline{g^x} f^x \, dx = \int \overline{f(x)} \hat{f}(x) \, dk$$

where $\overline{g^x}$ and $\overline{f^x}$ is the complex conjugate.